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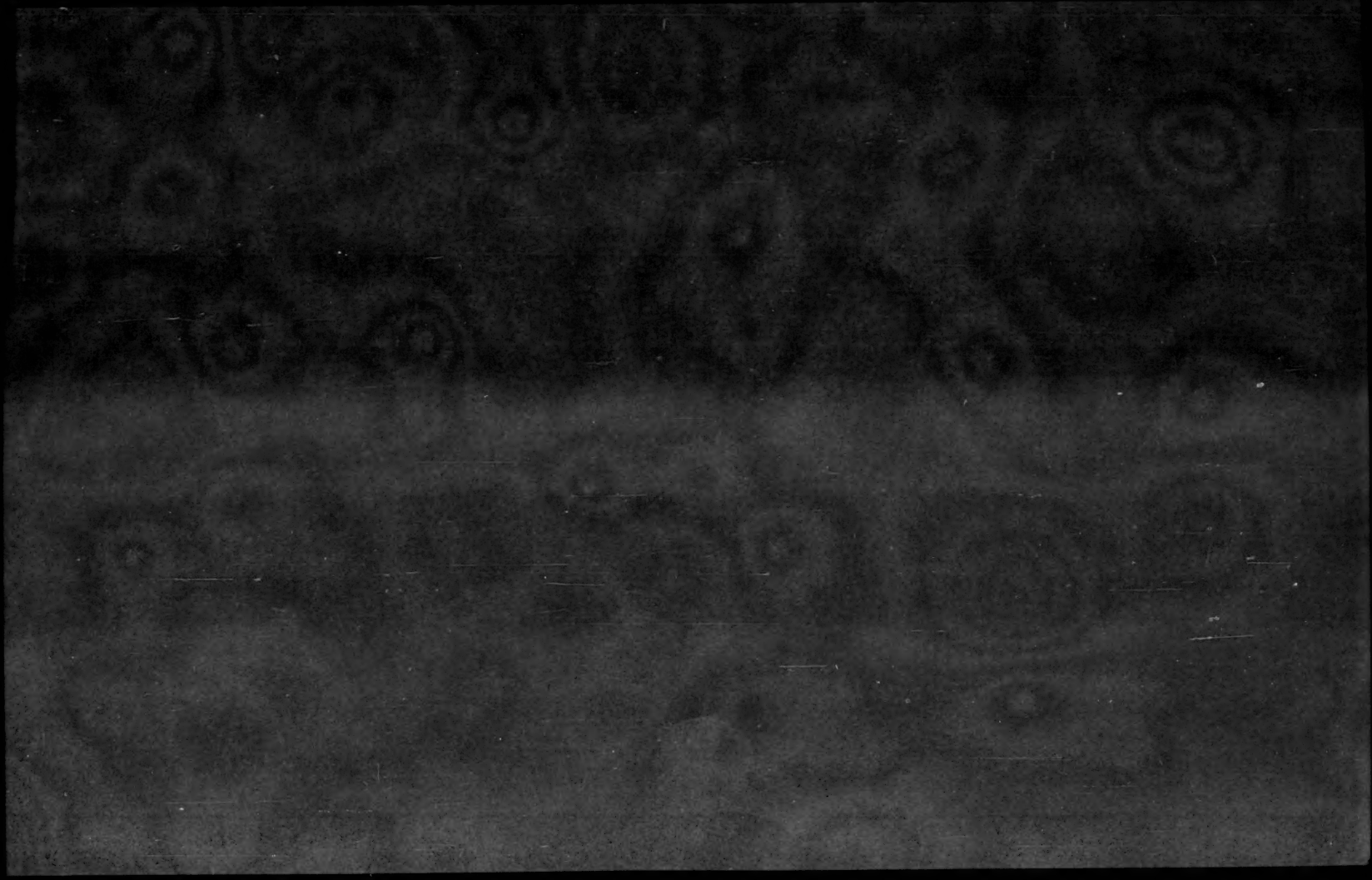

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SECTION THREE

Difference Methods in the Theory of Local Order Bases and Their Equivalent Normalized Function Bases

S. BEATTY, F.R.S.C.

1. The fundamental equation. We shall adopt

$$F \equiv u^n + u^{n-1}F_1 + \ldots + F_n = 0$$

as the fundamental algebraic equation, where

$$F_1, \ldots, F_n$$

denote rational functions of z. Furthermore, we shall suppose that F, considered as a polynomial in u, is either irreducible in the domain of rational functions of z with complex numbers as coefficients, or reducible but without any factor occurring a second time. The independent variable z will be restricted to the vicinity of $z = 0$, and the expansions of u will be taken as falling into r cycles, containing

$$\nu_1, \ldots, \nu_r$$

branches respectively and proceeding according to ascending powers of

$$\frac{1}{z^{\nu_1}}, \ldots, \frac{1}{z^{\nu_r}}$$

respectively, or their conjugates.

2. Order bases. By a local order basis

$$(\tau): \tau_1, \ldots, \tau_r$$

we shall mean a set of finite multiples of the orders

$$\frac{1}{\nu_1}, \ldots, \frac{1}{\nu_r}$$

respectively. Since we shall have only a single locality in mind, namely the vicinity of $z = 0$, we can afford to drop the word local and refer to (τ) simply as an order basis. A rational function of (z, u) not vanishing identically for any of the branches of our fundamental equation, say the rational function $R(z, u)$, furnishes in the orders which it takes for these branches an order basis

$$(\rho): \rho_1, \ldots, \rho_r.$$

In particular, $\partial F/\partial u$ is such a rational function of (z, u), and the order basis which it furnishes will be denoted by

$$(\mu): \mu_1, \ldots, \mu_r.$$

Order bases

$$(\tau): \tau_1, \ldots, \tau_r \text{ and } (\bar{\tau}): \bar{\tau}_1, \ldots, \bar{\tau}_r$$

will be said to be complementary to the level of $R(z, u)$, provided their sum $(\tau + \bar{\tau})$ is given by

$$\tau_1 + \bar{\tau}_1 = \rho_1 - 1 + \frac{1}{\nu_1}, \ldots, \tau_r + \bar{\tau}_r = \rho_r - 1 + \frac{1}{\nu_r}.$$

In the sequel, we shall use $\partial F/\partial u$ throughout as the rational function of (z, u) setting the level to be used and shall merely say that (τ) and $(\bar{\tau})$ are complementary, without always calling attention to the fact that the level function is understood to be $\partial F/\partial u$. In other words, when we say that (τ) and $(\bar{\tau})$ are complementary, what we mean is that $(\tau + \bar{\tau})$ is given by

$$\tau_1 + \bar{\tau}_1 = \mu_1 - 1 + \frac{1}{\nu_1}, \ldots, \tau_r + \bar{\tau}_r = \mu_r - 1 + \frac{1}{\nu_r}.$$

It is, of course, possible for a rational function of (z, u) to vanish identically for the branches of one or more cycles of our fundamental equation. In that case, the order of the rational function of (z, u) will be said to be infinite for each of the cycles in question.

We shall say that a rational function of (z, u) is built on an order basis (τ), provided its orders are all equal to or greater than the corresponding orders of (τ). Naturally, it is only the finite orders taken by the rational function of (z, u) that need concern us, since the infinite orders, if there are any such, are all greater than the corresponding orders of (τ). The totality of rational functions of (z, u) built on an order basis (τ) will be denoted by $R(z, u; (\tau))$. Evidently, there is one and only one rational function of (z, u) included in every totality $R(z, u; (\tau))$, and this is the function 0.

3. Function bases. Given n rational functions of (z, u)

$$(W): W_{n-1}, \ldots, W_0,$$

their associated determinant D has for rows the coefficients of $u^{n-1}, \ldots, 1$ in $W_{n-1}, \ldots, W_0$ taken one at a time and for columns the coefficients of $u^{n-1}, \ldots, 1$ taken one at a time in $W_{n-1}, \ldots, W_0$. Conversely, when D is given, we can write down (W) at once.

In order to guard against the possibility of a rational function of (z, u)

$$I_{n-1} W_{n-1} + \ldots + I_0 W_0$$

being the function 0, even though the rational functions of z

$$I_{n-1}, \ldots, I_0$$

are not all the function 0, we must stipulate that D is not the function 0. We shall then speak of (W) as a function basis for rational functions of (z, u) built on it and having the form

$$I_{n-1} W_{n-1} + \ldots + I_0 W_0,$$

in which the rational functions of z

$$I_{n-1}, \ldots, I_0$$

all have orders zero or more at $z = 0$. As already remarked, it is entirely a matter of orders in the vicinity of a single point, namely the point $z = 0$, and from here on we shall speak of the orders of $I_{n-1}, \ldots, I_0$ without always recalling that these are to be taken in the vicinity of $z = 0$. The totality of rational functions of (z, u) built on (W) will be denoted by $R(z, u; (W))$. Evidently, 0 is the only rational function of (z, u) included in every such totality.

The totality $R(z, u; (W'))$ is included under the totality $R(z, u; (W))$, provided the coefficients in the linear transformation expressing the n basis elements of (W') in terms of the n basis elements of (W) all have orders equal to or greater than zero. In that case, the determinant L' of this linear transformation has order equal to or greater than zero, which means that the order of D' is at least as great as that of D, seeing that $D' = L'D$.

Whenever the totalities $R(z, u; (W))$ and $R(z, u; (W'))$ are the same, we shall say that the function bases (W) and (W') are equivalent, or, in symbols, that $(W) \sim (W')$. In that case,

$$D' = L'D \text{ and } D = LD',$$

and so the determinants L' and L both have order zero, and the determinants L' and L are reciprocals.

A function basis (W) will be said to be normalized if the n basis elements

$$W_{n-1}, \ldots, W_0$$

are of degrees $n - 1, \ldots, 0$ in u. Obviously, we can go further and require the coefficients of the degree terms to be of the form

$$\left(\frac{1}{z}\right)^{\alpha_{n-1}}, \ldots, \left(\frac{1}{z}\right)^{\alpha_0},$$

which can always be achieved by dividing the basis elements through by the appropriate units, which are rational functions of z of order zero.

Given a normalized function basis (W), it is clear that the principal diagonal of its associated determinant D is made up of the terms

$$\left(\frac{1}{z}\right)^{\alpha_{n-1}}, \ldots, \left(\frac{1}{z}\right)^{\alpha_0},$$

while all the terms to the left of it are identically zero. It follows that the order of D is

$$-\sum_{j=0}^{n-1} \alpha_j.$$

4. Function bases equivalent to assigned order bases. An order basis (τ) and a function basis (W) are said to be equivalent, or, in symbols,

$$(\tau) \sim (W),$$

provided the totalities $R(z, u; (\tau))$ and $R(z, u; (W))$ are the same. Given an order basis (τ), it is easy to show the existence of a normalized function basis (W) equivalent to it. For, among the rational functions of (z, u) included under $R(z, u; (\tau))$ and of degree j in u, where j is any of $n-1, \ldots 0$, consider any one in which the coefficient of the degree term has the least possible order, and divide this one through by a rational function of z of order zero, so as to reduce the coefficient of the degree term to a single power of z. Denote by W_j the rational function of (z, u) obtained in this way. It is clear that the resulting normalized function basis

$$(W): W_{n-1}, \ldots, W_0$$

is equivalent to the assigned order basis (τ). It might be feared that there is no least possible order as spoken of above. However, by applying the Lagrange Interpolation Formula to the functions included under $R(z,u;(\tau))$, we soon see that all the coefficients of the n powers of u have orders never less than a fixed order, which can be stated once for all. Hence, given an order basis (τ), we can be sure of the existence of a function basis (W) equivalent to it. There is, of course, no thought of a converse here. That is, on being given a function basis (W), we cannot count on the existence of an order basis (τ) equivalent to it, since we know of cases where no such order basis (τ) exists. It suffices to give an example of such cases. Suppose u has no pole at $z = 0$. Then the exponents $\alpha_{n-1}, \ldots, \alpha_0$, as they appear in a function basis equivalent to an order basis (τ), must be in descending order, since u times any function in $R(z, u; (\tau))$ is itself in $R(z, u: (\tau))$. Hence, if $\alpha_{n-1}, \ldots, \alpha_0$, as they appear in (W), are not in descending order, there can be no order basis (τ) equivalent to (W), since all function bases equivalent to (τ) involve the same exponents $\alpha_{n-1}, \ldots, \alpha_0$ as (W) did. This leads to a contradiction, since at times we are saying that $\alpha_{n-1}, \ldots, \alpha_0$ are in descending order and again that they are not. However, this lack of a converse is not a matter of any importance, since the direction in which we plan to move is from (τ) to (W) — never the reverse.

5. Unit changes in (τ). Given an order basis

$$(\tau): \tau_1, \ldots, \tau_r,$$

and being in possession of a normalized function basis

$$(W): W_{n-1}, \ldots, W_0$$

equivalent to it, and proceeding to modify (τ) by either increasing or decreasing any single one of its orders $\tau_1, \ldots, \tau_r$ in the minimum amount

$$\frac{1}{\nu_1}, \ldots, \frac{1}{\nu_r}$$

respectively, we propose to examine how (W) must be modified in each case to give us a new normalized function basis equivalent to the new order basis. The changes in (τ) are spoken of as unit changes, since they either increase or decrease the sum

$$\sum_{t=1}^{r} \tau_t \nu_t$$

by unity.

No confusion will result from denoting by τ the particular one of the orders $\tau_1, \ldots, \tau_r$ chosen for the modification in the minimum amount $1/\nu$, so as to give either the increased order $\tau + 1/\nu$ or the decreased order $\tau - 1/\nu$. It also seems natural to denote the new order basis by $(\tau) + 1/\nu$ or $(\tau) - 1/\nu$, depending on whether the single order τ chosen for modification is either increased to $\tau + 1/\nu$ or decreased to $\tau - 1/\nu$.

It will be convenient to deal first with the case in which (τ) is increased to $(\tau) + 1/\nu$, after which the result can be applied to help deal with the case in which (τ) is decreased to $(\tau) - 1/\nu$. In other words, beginning with $(W) \sim (\tau)$, we have to deduce normalized function bases equivalent to $(\tau) + 1/\nu$ and $(\tau) - 1/\nu$ respectively.

6. The $(\tau) + 1/\nu$ case. Denoting a normalized function basis equivalent to (τ) by

$$(W): W_{n-1}, \ldots, W_0,$$

let us suppose that W_p is the last of its basis elements to have order τ for any particular branch of the cycle in question. Replace the basis elements $W_{n-1}, \ldots, W_{p+1}$ by the new basis elements

$$W_{n-1} + \lambda_{n-1} W_p, \ldots, W_{p+1} + \lambda_{p+1} W_p,$$

where the constants $\lambda_{n-1}, \ldots, \lambda_{p+1}$ are determined so that the new basis elements all have orders greater than τ for a particular branch of the cycle in question. It is, of course, clear that these constants are independent of the particular branch employed. This gives us then a normalized function basis

$$W_{n-1} + \lambda_{n-1} W_p, \ldots, W_{p+1} + \lambda_{p+1} W_p, \ldots, W_0$$

equivalent to both (W) and (τ). A function basis equivalent to $(\tau) + 1/\nu$ is, therefore,

$$W_{n-1} + \lambda_{n-1} W_p, \ldots, W_{p+1} + \lambda_{p+1} W_p, z W_p, \ldots, W_0,$$

which differs from what we just had through $z\ W_p$ taking the place of W_p.

We may sum up as follows what we have just shown:

THEOREM 1. *Given an order basis*

$$(\tau): \tau_1, \ldots, \tau_r,$$

and being in possession of a normalized function basis

$$(W): W_{n-1}, \ldots, W_0$$

equivalent to it, and supposing that W_p is the last of the basis elements of (W) to have order τ for the branches of the cycle for which $(\tau) + 1/\nu$ exceeds (τ) by the minimum amount $1/\nu$, we find that a normalized function basis equivalent to $(\tau) + 1/\nu$ is derived from (W) by replacing W_p by $z\ W_p$ and W_h by $W_h + \lambda_h\ W_p$ $(h = n-1, \ldots, p+1)$, where each λ_h is a unique constant chosen so that the corresponding basis element $W_h + \lambda_h\ W_p$ is of order greater than τ for the branches of the cycle for which $(\tau) + 1/\nu$ exceeds (τ).

This leads to two rather obvious corollaries:

COROLLARY 1.

$$\sum_{i=1}^{r} \tau_i \nu_i + \sum_{j=0}^{n-1} \alpha_j = \sum_{i=1}^{r} \tau'_i \nu_i + \sum_{j=0}^{n-1} \alpha'_j$$

for any pair of order bases (τ), (τ') and their associated exponents (α), (α').

COROLLARY 2. *There are*

$$\frac{n!}{\nu_1! \ldots \nu_r!}$$

ways of passing by minimum increases of order from (τ) to $(\tau + 1)$, but they all convert the normalized function basis

$$W_{n-1}, \ldots, W_0$$

equivalent to (τ) into the normalized function basis

$$z\ W_{n-1}, \ldots, z\ W_0$$

equivalent to $(\tau + 1)$.

Only the first of these calls for any comment. It is a simple consequence of a result established in **(2)**, to the effect that

$$\sum_{i=1}^{r} \tau_i \nu_i + \sum_{j=0}^{n-1} \alpha_j = \frac{1}{2} \sum_{i=1}^{r} \left(\mu_i - 1 + \frac{1}{\nu_i} \right) \nu_i.$$

We shall, however, deduce it from Theorem 1 above. We shall denote the left and right sides of the equation to be established by $S(\tau)$ and $S(\tau')$

respectively. Consider an example of a single minimum increase in order, such, indeed, as is involved in passing from order basis (τ) to order basis $(\tau') = (\tau) + 1/\nu$. We see at once that

$$\sum_{i=1}^{r} \tau_i' \nu_i = \sum_{i=1}^{r} \tau_i \nu_i + 1, \sum_{j=0}^{n-1} \alpha_j' = \sum_{j=0}^{n-1} \alpha_j - 1,$$

which means that

$$S(\tau) = S(\tau').$$

Consider also an example of a single minimum decrease in order, such as is involved in passing from order basis (τ) to order basis $(\tau') = (\tau) - 1/\nu$. If the sense of the ordering is reversed, it becomes a matter of passing from order basis (τ') to order basis $(\tau) = (\tau') + 1/\nu$, and again this means that $S(\tau) = S(\tau')$. But, we can pass from any order basis (τ) to any other order basis (τ') by a succession of single minimum increases and decreases. It follows, therefore, for any pair of order bases (τ), (τ') that $S(\tau) = S(\tau')$.

7. The $(\tau) - 1/\nu$ case. The treatment to be given for this case is based on a first application of the multiplication technique defined in **(2)**, followed by an application of Theorem 1, and concluded by a second application of the multiplication technique.

The multiplication technique begins with a normalized function basis

$$(W): W_{n-1}, \ldots, W_0$$

equivalent to an order basis

$$(\tau): \tau_1, \ldots, \tau_r$$

and ends up by supplying a normalized function basis

$$(\bar{W}): \bar{W}_{n-1}, \ldots, \bar{W}_0$$

equivalent to the complementary order basis

$$(\bar{\tau}): \bar{\tau}_1, \ldots, \bar{\tau}_r.$$

The technique consists in requiring that the coefficient of u^{n-1} in the reduced form of each of the n^2 products

$$W_h \bar{W}_k \ (h = 0, \ldots, n-1; k = 0, \ldots, n-1)$$

should have an order equal to or greater than zero, or effectively that the coefficient of u^{n-1} in the reduced form of each of the $n(n+1)/2$ products

$$W_h \bar{W}_k \ (h = n-k-1, \ldots, n-1; k = 0, \ldots, n-1)$$

should have an order equal to or greater than zero. In other words, for each value of k taken from $0, \ldots, n-1$ we are concerned with the $k+1$ products

$$W_{n-k-1} \bar{W}_k, \ldots, W_{n-1} \bar{W}_k.$$

We make use of the first, . . . , last of these products to determine the coefficients of $u^k, \ldots, u^0$ respectively in $\bar{W}_k$. In particular, the coefficient of u^k is seen to be $z^{\alpha_{n-k-1}}$, which is the reciprocal of the coefficient of u^{n-k-1} in W_{n-k-1}. This means that the coefficients of the degree terms in the basis functions of

$$(\bar{W}): \bar{W}_{n-1}, \ldots, \bar{W}_0$$

are the powers

$$z^{\alpha_0}, \ldots, z^{\alpha_{n+1}}$$

respectively, which corresponds to the fact that the coefficients of the degree terms in the basis functions of

$$(W): W_{n-1}, \ldots, W_0$$

are the powers

$$\left(\frac{1}{z}\right)^{\alpha_{n-1}}, \ldots, \left(\frac{1}{z}\right)^{\alpha_0}$$

respectively.

The sequence to be followed in the ensuing discussion is to pass from

$$(W) \sim (\tau) \text{ to } (\bar{W}) \sim (\bar{\tau})$$

by the multiplication technique, then from

$$(\bar{W}) \sim (\bar{\tau}) \text{ to } (\bar{W}') \sim (\bar{\tau}) + \frac{1}{\nu}$$

by Theorem 1, and finally from

$$(\bar{W}') \sim (\bar{\tau}) + \frac{1}{\nu} \text{ to } (W') \sim (\tau) - \frac{1}{\nu}$$

by the multiplication technique.

In the normalized function basis $(\bar{W}) \sim (\bar{\tau})$ obtained by applying the multiplication technique to the normalized function basis $(W) \sim (\tau)$, let us suppose that $\bar{W}_q$ is the last of the basis elements

$$\bar{W}_{n-1}, \ldots, \bar{W}_0$$

to have order $\bar{\tau}$ for the branches of the cycle in question. According to Theorem 1, we can say that the normalized function basis $(\bar{W}') \sim (\bar{\tau}) + 1/\nu$ is given by

$$\bar{W}_{n-1} + \bar{\lambda}_{n-1}\bar{W}_q, \ldots, \bar{W}_{q+1} + \lambda_{q+1}\bar{W}_q, z\bar{W}_q, \ldots, \bar{W}_0,$$

in which $\bar{\lambda}_{n-1}, \ldots, \bar{\lambda}_{q+1}$ are unique constants chosen so that the functions

$$\bar{W}_{n-1} + \bar{\lambda}_{n-1}\bar{W}_q, \ldots, \bar{W}_{q+1} + \bar{\lambda}_{q+1}\bar{W}_q$$

all have orders greater than $\bar{\tau}$ for the branches of the cycle in question.

It only remains to apply the multiplication technique to the normalized function basis $(\bar{W}') \sim (\bar{\tau}) + 1/\nu$ to obtain the normalized function basis $(W') \sim (\tau) - 1/\nu$. The $n - q - 1$ basis functions

$$W_{n-q-2}, \ldots, W_0$$

occurring in $(W) \sim (\tau)$ all satisfy the multiplication technique relative to the n basis functions of $(\bar{W}) \sim (\bar{\tau})$. But, these same $n - q - 1$ basis functions all satisfy the multiplication technique relative to the n basis functions of $(\bar{W}') \sim (\bar{\tau}) + 1/\nu$, for the only difference is that $\bar{W}_q$ in the former case gives place to $z\,\bar{W}_q$ in the latter and similarly $\bar{W}_h$ gives place to $\bar{W}_h + \bar{\lambda}_h\,\bar{W}_q$ for $h = q+1, \ldots, n-1$, all of which produces no change whatever in the coefficients of u^{n-1} in the reduced forms of the products.

The basis function W_{n-q-1} serving in $(W) \sim (\tau)$ does not serve in $(W') \sim (\tau) - 1/\nu$, but the function

$$\Delta = \frac{W_{n-q-1} + \theta_{n-q-2}W_{n-q-2} + \ldots + \theta_0 W_0}{z}$$

serves instead, where it remains to indicate how the constants $\theta_{n-q-2}, \ldots, \theta_0$ are to be chosen. It is a matter of dealing with the products

$$\Delta(z\bar{W}_q) \text{ and } \Delta(\bar{W}_h + \bar{\lambda}_h\bar{W}_q) \text{ for } h = q+1, \ldots, n-1.$$

For the first of these, all that we need consider is

$$\frac{W_{n-q-1}}{z}(z\bar{W}_q),$$

in which the coefficient of u^{n-1} is unity. For the typical product $\Delta\,(\bar{W}_h + \bar{\lambda}_h\,\bar{W}_q)$, all that we need consider is

$$\frac{W_{n-q-1} + \ldots + \theta_{n-h-1}W_{n-h-1}}{z}(\bar{W}_h + \bar{\lambda}_h\bar{W}_q),$$

which can be further shortened to

$$\frac{W_{n-q-1} + \ldots + \theta_{n-h}W_{n-h}}{z}(\bar{W}_h + \bar{\lambda}_h\bar{W}_q) + \theta_{n-h-1}\frac{W_{n-h-1}}{z}(\bar{W}_h),$$

on leaving out items contributing nothing to the coefficient of u^{n-1} in the reduced form of the product. The first part of this when reduced is of the form

$$\left(\frac{\theta}{z} + \ldots\right)u^{n-1} + \ldots,$$

where θ depends on the constants

$$\theta_{n-q-2}, \ldots, \theta_{n-h},$$

supposed known already, while the second part is

$$\left(\frac{\theta_{n-h-1}}{z}\right)u^{n-1} + \ldots.$$

All that is necessary is to choose $\theta_{n-h-1} = -\theta$. This means that we now know the exact form of Δ.

It now follows that Δ in $(W') \sim (\tau) - 1/\nu$ has the order $\tau - 1/\nu$ for

the branches of the cycle in question, since the corresponding basis function in $(W) \sim (\tau)$ is

$$W_{n-q-1} = z\,\Delta - \theta_{n-q-2}\,W_{n-q-2} - \ldots - \theta_0\,W_0.$$

Evidently, W_{n-q-1} as a basis function in $(W) \sim (\tau)$ could be replaced by $z\,\Delta$, seeing that this merely involves adding on to W_{n-q-1} certain constant multiples of the basis functions $W_{n-q-2}, \ldots, W_0$ of lower degrees.

Finally, it is plain that

$$W_{n-1}, \ldots, W_{n-q},$$

which serve as basis functions of $(W) \sim (\tau)$, will also serve as basis functions of $(W') \sim (\tau) - 1/\nu$, but, of course, none of them has the order $\tau - 1/\nu$ for the cycle in question. Naturally,

$$W_{n-1} + \theta_{n-1}\,\Delta, \ldots, W_{n-q} + \theta_{n-q}\,\Delta, \text{ where}$$

$\theta_{n-1}, \ldots, \theta_{n-q}$ are arbitrary constants, will serve equally well as basis functions, and any of these will have the order $\tau - 1/\nu$ for the cycle in question according as the arbitrary constant appearing in it is given a value different from zero. On the whole, it is the simplest and most natural to choose the constants $\theta_{n-1}, \ldots, \theta_{n-q}$ all zero.

What we have established may now be stated as follows:

THEOREM 2. *Where $\bar{W}_q$ is the last of the basis functions in $(\bar{W}) \sim (\bar{\tau})$ to have order $\bar{\tau}$ for the cycle in question, and where*

$$\bar{\lambda}_{n-1}, \ldots, \bar{\lambda}_{q+1}$$

are determined so that the functions

$$\bar{W}_{n-1} + \bar{\lambda}_{n-1}\,\bar{W}_q, \ldots, \bar{W}_{q+1} + \bar{\lambda}_{q+1}\,\bar{W}_q$$

all have orders greater than $\bar{\tau}$ for the cycle in question, a function basis $(W') \sim (\tau) - 1/\nu$ can be obtained from the function basis $(W) \sim (\tau)$ by replacing the basis function W_{n-q-1} occurring in it by the basis function

$$\Delta = \frac{W_{n-q-1} + \theta_{n-q-2}\,W_{n-q-2} + \ldots + \theta_0\,W_0}{z},$$

where the constants

$$\theta_{n-q-2}, \ldots, \theta_0$$

are determined by requiring that the coefficients of u^{n-1} in the reduced forms of the products

$$\Delta(\bar{W}_h + \bar{\lambda}_h\,\bar{W}_q) \text{ for } h = q+1, \ldots, n-1$$

all have orders zero or more.

We now have the following corollaries:

COROLLARY 3. *The product of the last of the basis functions in $(\bar{W}) \sim (\bar{\tau})$ to have order $\bar{\tau}$ for the cycle in question by the last of the basis functions in*

$(W') \sim (\tau) - 1/v$ *to have order* $\tau - 1/v$ *for the cycle in question is of the form*

$$\frac{u^{n-1}}{z} + \dots,$$

and this without reduction being necessary.

COROLLARY 4

$$\sum_{i=1}^{r} \tau_i \nu_i + \sum_{j=0}^{n-1} \alpha_j = \frac{1}{2} \sum_{i=1}^{r} \left(\mu_i - 1 + \frac{1}{\nu_i} \right) \nu_i.$$

It is enough to indicate how this follows from Corollary 1 and the equations

$$\bar{\alpha}_j + \alpha_{n-j-1} = 0 \qquad j = 0, \dots, n-1.$$

Twice the left side of the equation of Corollary 4 is

$$\sum_{i=1}^{r} (\tau_i + \bar{\tau}_i) \nu_i + \sum_{j=0}^{n-1} (\alpha_j + \bar{\alpha}_j),$$

and of these two sums the first is

$$\sum_{i=1}^{r} \left(\mu_i - 1 + \frac{1}{\nu_i} \right) \nu_i,$$

while the second is zero.

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SECTION THREE

The Bianchi Identities for the Conformal Curvature Tensor*

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1. Introduction. It is known that the number of the fundamental equations which determine the local and isometrical imbedding of a Riemannian Space in a Euclidean Space can be reduced on account of the "ordinary" Bianchi Identities **(2)**. The author has found recently that certain linear combinations of the covariant derivatives of the *conformal* curvature tensor, which are identically zero, play a similar role in the imbedding of a Riemannian Space in a conformally Euclidean Space. These identities are implicitly contained in some earlier works (e.g. **(3)** pp. 90-1), but have, so far as I know, not been published in their explicit form. Being conformally invariant, they are sufficiently interesting in themselves to deserve a separate note.

As could be expected the "Conformal" Bianchi Identities can be expressed as linear combinations of the ordinary Bianchi Identities. This will enable us to determine their number which is important in the investigation of imbedding properties of Riemannian Spaces in conformally Euclidean Spaces.

2. The conformal Bianchi Identities. Let

$$C^h_{ijk} = R^h_{ijk} + \frac{1}{n-2}(\delta^h_j R_{ik} - \delta^h_k R_{ij} + g_{ik} R^h_j - g_{ij} R^h_k) + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \tag{1}$$

be the conformal curvature tensor of a V_n ($n \geqslant 4$) (see **(3)**, pp. 90–1, from which all the notation conventions have been adopted).

Let B^h_{ijkl} denote the (identically zero) Bianchi tensor, i.e.:

$$B^h_{ijkl} = R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k} \tag{2}$$

If (1) is covariantly differentiated with respect to the index l and then the indices j, k, l cyclically permuted we obtain because of (2):

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$$C^h{}_{ijk,l} + C^h{}_{ikl,j} + C^h{}_{ilj,k} = B^h{}_{ijkl}$$

$$(3) \qquad + \frac{1}{n-2}(\delta^h_j R_{ikl} + \delta^h_k R_{ilj} + \delta^h_l R_{ijk}$$

$$- g_{ij} R^h{}_{kl} - g_{ik} R^h{}_{lj} - g_{il} R^h{}_{jk})$$

where ((3), pp. 90–1)

$$(4) \quad R_{ijk} = R_{ij,k} - R_{ik,j} + \frac{1}{2(n-1)}(g_{ik} R_{,j} - g_{ij} R_{,k}),$$

$$(5) \quad R^h{}_{jk} = g^{hi} R_{ijk} = R^h{}_{j,k} - R^h{}_{k,j} + \frac{1}{2(n-1)}(\delta^h_k R_{,j} - \delta^h_j R_{,k}).$$

If we put in (2) $l = h$ and sum, we obtain

$$B^h{}_{ijkh} = R^h{}_{ijk,h} + R_{ik,j} - R_{ij,k}$$

or, multiplying by g^{ij} and summing over i, j:

$$(6) \qquad g^{ij} B^h{}_{ijkh} = 2R^l_{k,l} - R_{,k}$$

On the other hand we obtain from (5):

$$(7) \qquad R^h{}_{kh} = R^h{}_{k,h} - \tfrac{1}{2} R_{,k}.$$

Hence, on account of (6)

$$(8) \qquad R^h{}_{kh} = \tfrac{1}{2} g^{pq} B^h{}_{pqkh}.$$

We return now to formula (3) and contract for h, l. Making use of (8) and the fact that $C_{ij} = 0$, we obtain:

$$(9) \qquad C^h{}_{ijk,h} = \frac{n-3}{n-2} R_{ijk} + B^h{}_{ijkh}$$

$$- \frac{g^{pq}}{2(n-2)}(g_{ij} B^h{}_{pqkh} - g_{ik} B^h{}_{pqjh}).$$

This formula reduces, of course, to (28, 16) in (3) if we put $B^h{}_{ijkl} = 0$.

We now introduce the (identically zero) tensor B_{ijk} by:

$$(10) \qquad B_{ijk} = B^h{}_{ijkh} - \frac{g^{pq}}{2(n-2)}(g_{ij} B^h{}_{pqkh} - g_{ik} B^h{}_{pqjh})$$

and solve (9) with respect to R_{ijk}:

$$(11) \qquad R_{ijk} = \frac{n-2}{n-3}(C^h{}_{ijk,h} - B_{ijk})$$

This assumes, of course, that $n \neq 3$ but this does not restrict the generality of our discussion since for $n = 3$ the conformal curvature tensor $C^h{}_{ijk}$ does not exist ((3), pp. 90–1).

[That this is so can also be seen from the following consideration: The number of independent components of $C^h{}_{ijk}$ equals the number of components of $R^h{}_{ijk}$ minus the number of components of C_{ij}, which, being identically equal to zero, represent linear relations between the $C^h{}_{ijk}$.

Thus:

$$\text{Number of } C^h{}_{ijk} = n^2(n^2-1)/12 - n(n+1)/2 = \\ = n(n+1)\,(n+2)\,(n-3)/12,$$

from which it follows that for $n = 3$, $C^h{}_{ijk}$ does not exist.]

We obtain similarly from (5):

$$R^h{}_{jk} = \frac{n-2}{n-3} g^{hi}(C^p{}_{ijk,p} - B_{ijk}). \tag{12}$$

If (11) and (12) are substituted into (3) and the C's and the B's separated, we obtain finally:

$$\begin{aligned} &C^h{}_{ijk,l} + C^h{}_{ikl,j} + C^h{}_{ilj,k} \\ &- \frac{g^{hq}}{n-3}(g_{qj} C^p{}_{ikl,p} + g_{qk} C^p{}_{ilj,p} + g_{ql} C^p{}_{ijk,p} \\ &- g_{ij} C^p{}_{qkl,p} - g_{ik} C^p{}_{qlj,p} - g_{il} C^p{}_{qjk,p}) \\ &\equiv B^h{}_{ijkl} - \frac{g^{hq}}{n-3}(g_{qj} B_{ikl} + g_{qk} B_{ilj} + g_{ql} B_{ijk} \\ &- g_{ij} B_{qkl} - g_{ik} B_{qlj} - g_{il} B_{qjk}) \equiv 0. \end{aligned} \tag{13}$$

We have therefore the Theorem:

In any Riemannian Space V_n ($n \geqslant 4$) the following relations, involving the covariant derivatives of the conformal curvature tensor, are satisfied identically:

$$\begin{aligned} K^h{}_{ijkl} \equiv\ & C^h{}_{ijk,l} + C^h{}_{ikl,j} + C^h{}_{ilj,k} \\ &- \frac{g^{hq}}{n-3}(g_{qj} C^p{}_{ikl,p} + g_{qk} C^p{}_{ilj,p} + g_{ql} C^p{}_{ijk,p} \\ &- g_{ij} C^p{}_{qkl,p} - g_{ik} C^p{}_{qlj,p} - g_{il} C^p{}_{qjk,p}) \equiv 0. \end{aligned} \tag{I}$$

We shall call (I) the "Conformal" Bianchi Identities. Their left-hand sides, denoted by $K^h{}_{ijkl}$, (which can be called the Conformal Bianchi Tensor), are, according to (13) linear combinations of the ordinary Bianchi Identities. It is obvious that the $K^h{}_{ijkl}$ are conformally invariant.

If in (I) the index h is lowered we obtain an equivalent set of identities:

$$\begin{aligned} K_{hijkl} \equiv\ & C_{hijk,l} + C_{hikl,j} + C_{hilj,k} \\ &- \frac{1}{n-3}(g_{hj} C^p{}_{ikl,p} + g_{hk} C^p{}_{ilj,p} + g_{hl} C^p{}_{ijk,p} \\ &- g_{ij} C^p{}_{hkl,p} - g_{ik} C^p{}_{hlj,p} - g_{il} C^p{}_{hjk,p}) \equiv 0. \end{aligned} \tag{II}$$

It follows from (13) that

$$\begin{aligned} K_{hijkl} \equiv B_{hijkl} - \frac{1}{n-3}&(g_{hj} B_{ikl} + g_{hk} B_{ilj} + g_{hl} B_{ijk} \\ &- g_{ij} B_{hkl} - g_{ik} B_{hlj} - g_{il} B_{hjk}) \equiv 0. \end{aligned} \tag{14}$$

3. The number of the conformal Bianchi Identities. This number equals the number of independent components of the tensor $K^h{}_{ijkl}$ or, equivalently, of K_{hijkl}. By (14) it follows that K_{hijkl} will satisfy the same linear relations as the tensor B_{hijkl}. If we contract in (I) for h and l we obtain:

$$K^h{}_{ijkh} = C^h{}_{ijk,h} - \frac{1}{n-3}(n-3)\, C^p{}_{ijk,p} \equiv 0$$

i.e. the tensor $K^h{}_{ijkl}$ satisfies, in addition, the linear relations:

$$K_{ijk} \equiv K^p{}_{ijkp} \equiv g^{pq} K_{pijkq} \equiv 0. \tag{15}$$

The number of these relations is $n(n^2-1)/3$ (see **(2)**, p. 158. Because of (14) and (15) the number of independent components of K_{hijkl} will be equal to the number of components of B_{hijkl} minus the number of those of K_{ijk}, i.e. (see **(1)**):

$$\begin{aligned}\text{Number of conformal Bianchi Identities} &= \\ &= n^2(n^2-1)\,(n-2)/24 - n\,(n^2-1)/3 = \\ &= n(n^2-1)\,(n+2)\,(n-4)/24.\end{aligned}$$

It follows therefore that the conformal Bianchi Identities are non-trivial only for $n \geqslant 5$.

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SECTION THREE

Sous-algèbres de Cartan et décompositions de Levi dans les algèbres de Lie

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Presented by I. HALPERIN, F.R.S.C.

Soit $\mathfrak{a}$ une algèbre de Lie sur un corps de caractéristique 0. Les sous-algèbres de Cartan de $\mathfrak{a}$ sont définies par Chevalley **(1)** comme les sous-algèbres nilpotentes de $\mathfrak{a}$ égales à leur normalisateur. Nous allons compléter sur quelques points les propriétés, établies dans **(1)**, de ces sous-algèbres.

1. Décomposition d'une sous-algèbre de Cartan

PROPOSITION 1. *Soient $\mathfrak{a}$ une algèbre de Lie sur un corps de caractéristique 0, $\mathfrak{r}$ son radical, $\mathfrak{s}$ une section de Levi de $\mathfrak{a}$.*

(i) *Soit $\mathfrak{h}$ une sous-algèbre de Cartan de $\mathfrak{a}$, somme d'une sous-algèbre de Cartan $\mathfrak{k}$ de $\mathfrak{s}$ et d'une sous-algèbre $\mathfrak{l}$ de $\mathfrak{r}$. Alors, $[\mathfrak{l}, \mathfrak{k}] = 0$, et $\mathfrak{l}$ est une sous-algèbre de Cartan du centralisateur de $\mathfrak{k}$ dans $\mathfrak{r}$.*

(ii) *Réciproquement, soient $\mathfrak{k}$ une sous-algèbre de Cartan de $\mathfrak{s}$, $\mathfrak{c}$ le centralisateur de $\mathfrak{k}$ dans $\mathfrak{r}$, $\mathfrak{l}$ une sous-algèbre de Cartan de $\mathfrak{c}$. Alors, $\mathfrak{h} = \mathfrak{k} + \mathfrak{l}$ est une sous-algèbre de Cartan de $\mathfrak{a}$.*

Démonstration. (i) Soit $\mathfrak{h}$ une sous-algèbre de Cartan de $\mathfrak{a}$, somme d'une sous-algèbre de Cartan $\mathfrak{k}$ de $\mathfrak{s}$ et d'une sous-algèbre $\mathfrak{l}$ de $\mathfrak{r}$. Considérons la représentation $x \to ad_{\mathfrak{a}} x$ de l'algèbre semi-simple $\mathfrak{s}$ dans l'espace $\mathfrak{a}$. Pour $x \in \mathfrak{k}$, $ad_{\mathfrak{a}} x$ est semi-simple. Or, $\mathfrak{l}$ est un idéal de $\mathfrak{h}$, donc $\mathfrak{l}$ est stable pour $ad_{\mathfrak{a}} x$, et la restriction u_x de $ad_{\mathfrak{a}} x$ à $\mathfrak{l}$ est semi-simple. Comme $\mathfrak{h}$ est une algèbre nilpotente, u_x est nilpotent. Donc u_x est nul, c'est-à-dire que $\mathfrak{l}$ est contenu dans le centralisateur $\mathfrak{c}$ de $\mathfrak{k}$ dans $\mathfrak{r}$. Maintenant, soit y un élément de $\mathfrak{c}$ appartenant au normalisateur de $\mathfrak{l}$ dans $\mathfrak{c}$, c'est-à-dire tel que $[\mathfrak{l}, y] \subset \mathfrak{l}$. On a $[\mathfrak{h}, y] = [\mathfrak{l}, y] + [\mathfrak{k}, y] = [\mathfrak{l}, y] \subset \mathfrak{l} \subset \mathfrak{h}$; comme $\mathfrak{h}$ est sous-algèbre de Cartan de $\mathfrak{a}$, on a $y \in \mathfrak{h}$, donc $y \in \mathfrak{h} \cap \mathfrak{r} = \mathfrak{l}$. Ceci prouve que $\mathfrak{l}$ est sous-algèbre de Cartan de $\mathfrak{c}$.

(ii) Soient $\mathfrak{k}$ une sous-algèbre de Cartan de $\mathfrak{s}$, $\mathfrak{c}$ le centralisateur de $\mathfrak{k}$ dans $\mathfrak{r}$, $\mathfrak{l}$ une sous-algèbre de Cartan de $\mathfrak{c}$, $\mathfrak{h}$ l'algèbre nilpotente $\mathfrak{k} + \mathfrak{l}$. Soit $x = y + z$ ($y \in \mathfrak{s}$, $z \in \mathfrak{r}$) un élément du normalisateur de $\mathfrak{h}$ dans $\mathfrak{a}$. Pour $u \in \mathfrak{k}$, on a $[y, u] + [z, u] = [x, u] \in \mathfrak{h}$, et $[y, u] \in \mathfrak{s}$, $[z, u] \in \mathfrak{r}$, donc $[y, u] \in \mathfrak{k}$. Donc $[y, \mathfrak{k}] \subset \mathfrak{k}$. Puisque $\mathfrak{k}$ est son propre normalisateur dans $\mathfrak{s}$, on a $y \in \mathfrak{k}$. Alors, z appartient au normalisateur de $\mathfrak{h}$ dans $\mathfrak{a}$. Pour $t \in \mathfrak{k}$,

on a $[t, z] \in \mathfrak{h}$ et $[t, z] \in \mathfrak{r}$, donc $[t, z] \in \mathfrak{l} \subset \mathfrak{c}$, donc $(ad_{\mathfrak{a}} t)^2 z = 0$. Comme $ad_{\mathfrak{a}} t$ est semi-simple, on en conclut que $(ad_{\mathfrak{a}} t)z = 0$, c'est-à-dire que $z \in \mathfrak{c}$. Enfin, on a $[z, \mathfrak{l}] \subset \mathfrak{h}$, et $[z, \mathfrak{l}] \subset \mathfrak{r}$, donc $[z, \mathfrak{l}] \subset \mathfrak{l}$; comme $\mathfrak{l}$ est son propre normalisateur dans $\mathfrak{c}$, on a $z \in \mathfrak{l} \subset \mathfrak{h}$, donc $x \in \mathfrak{h}$, ce qui prouve que $\mathfrak{h}$ est une sous-algèbre de Cartan de $\mathfrak{a}$.

2. Unicité de la décomposition

LEMME 1. *Soient $\mathfrak{a}$ une algèbre de Lie sur un corps de caractéristique 0, $\mathfrak{r}$ son radical, $\mathfrak{q} = [\mathfrak{a}, \mathfrak{r}]$ son radical nilpotent, $\mathfrak{s}$ une section de Levi de $\mathfrak{a}$, $\mathfrak{k}$ une sous-algèbre de Cartan de $\mathfrak{s}$, $\mathfrak{c}$ son centralisateur dans $\mathfrak{r}$. Soit x un élément de $\mathfrak{q}$, et u l'automorphisme spécial* $\exp ad_{\mathfrak{a}} x$. *Si $u(\mathfrak{k}) \subset \mathfrak{k} + \mathfrak{c}$, on a $x \in \mathfrak{c}$.*

Démonstration. Soit $C^0\mathfrak{q} = \mathfrak{q}$, $C^1\mathfrak{q}$, $C^2\mathfrak{q}, \ldots, C^{n-1}\mathfrak{q} \neq 0$, $C^n\mathfrak{q} = 0$ la suite centrale descendante de $\mathfrak{q}$ (rappelons que $C^{i+1}\mathfrak{q} = [\mathfrak{q}, C^i\mathfrak{q}]$). Considérons la représentation adjointe ρ de $\mathfrak{s}$ dans $\mathfrak{q}$. Elle est semi-simple, et les $C^i\mathfrak{q}$, qui sont stables pour toute dérivation de $\mathfrak{q}$, sont stables pour ρ. Soit $\mathfrak{q}_i$ un supplémentaire de $C^i\mathfrak{q}$ dans $C^{i-1}\mathfrak{q}$ stable pour ρ $(i = 1, 2, \ldots, n)$. La restriction ρ_i de ρ à $\mathfrak{q}_i$ est semi-simple, et ρ_i définit une représentation semi-simple σ_i de $\mathfrak{k}$ dans $\mathfrak{q}_i$. Soit $\mathfrak{q}_i'$ l'ensemble des éléments de $\mathfrak{q}_i$ annulés par σ_i, c'est-à-dire le centralisateur de $\mathfrak{k}$ dans $\mathfrak{q}_i$, et $\mathfrak{q}_i''$ un supplémentaire de $\mathfrak{q}_i'$ dans $\mathfrak{q}_i$ stable pour σ_i. Soit $x = x_1' + x_1'' + x_2' + x_2'' + \ldots + x_n' + x_n''$, où $x_i' \in \mathfrak{q}_i'$, $x_i'' \in \mathfrak{q}_i''$. Raisonnant par l'absurde, supposons les x_i'' non tous nuls; on aurait par exemple $x_1'' = x_2'' = \ldots = x_{p-1}'' = 0$, $x_p'' \neq 0$. Pour $h \in \mathfrak{k}$, on a

$$(ad\, x) \cdot h = \sum_{i=1}^{n} [x_i'', h] \in [x_p'', h] + C^p\mathfrak{q} \subset C^{p-1}\mathfrak{q}$$

donc $(ad\, x)^k \cdot h \in C^p\mathfrak{q}$ pour $k \geqslant 2$, et par suite

$$u(h) = h + [x_p'', h] + y$$

avec $y \in C^p\mathfrak{q}$. Par hypothèse, $u(h) \in \mathfrak{k} + \mathfrak{c}$. Donc $[x_p'', h] + y$ appartient au centralisateur de $\mathfrak{k}$ dans $C^{p-1}\mathfrak{q}$, c'est-à-dire à $\mathfrak{q}_p' + \mathfrak{q}_{p+1}' + \ldots + \mathfrak{q}_n'$. Donc $[h, x_p''] \in \mathfrak{q}_p'$. Mais, comme $x_p'' \in \mathfrak{q}_p''$, on a $[h, x_p''] \in \mathfrak{q}_p''$. Donc $[h, x_p''] = 0$ et par suite $[\mathfrak{k}, x_p''] = 0$. Donc $x_p'' \in \mathfrak{q}_p'$ et finalement $x_p'' = 0$, ce qui est contradictoire. Ceci prouve que $x = x_1' + x_2' + \ldots + x_n' \in \mathfrak{c}$.

PROPOSITION 2. *Soient $\mathfrak{a}$ une algèbre de Lie sur un corps de caractéristique 0, $\mathfrak{r}$ son radical, $\mathfrak{h}$ une sous-algèbre de Cartan de $\mathfrak{a}$. Alors, $\mathfrak{h}$ est, d'une manière et d'une seule, somme de $\mathfrak{h} \cap \mathfrak{r}$ et d'une sous-algèbre de Cartan d'une section de Levi de $\mathfrak{a}$.*

Démonstration. L'existence de la décomposition est une partie de la prop. 20, p. 222, de **(1)**. Prouvons l'unicité. Soient $\mathfrak{s}$ et $\mathfrak{s}'$ deux sections de Levi de $\mathfrak{a}$ telles que $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{s}) + (\mathfrak{h} \cap \mathfrak{r}) = (\mathfrak{h} \cap \mathfrak{s}') + (\mathfrak{h} \cap \mathfrak{r})$. Soit x un élément du radical nilpotent de $\mathfrak{a}$ tel que $u = \exp ad_{\mathfrak{a}} x$ transforme $\mathfrak{s}$ en $\mathfrak{s}'$ (th. de Levi-Malcev). Etant donné un élément de $\mathfrak{s}$, son transformé par

u est l'unique élément de $\mathfrak{s}'$ qui lui est congru modulo $\mathfrak{r}$. Or, pour $h \in \mathfrak{h} \cap \mathfrak{s}$, il existe un élément de $\mathfrak{h} \cap \mathfrak{s}'$ congru à h modulo $\mathfrak{h} \cap \mathfrak{r}$; donc $u(h) \in \mathfrak{h} \cap \mathfrak{s}'$ et finalement $u(h) - h \in \mathfrak{h} \cap \mathfrak{r}$. D'après la prop. 1(i), $\mathfrak{h} \cap \mathfrak{r}$ est contenu dans le centralisateur $\mathfrak{c}$ de $\mathfrak{h} \cap \mathfrak{s}$ dans $\mathfrak{r}$. On voit que $u(\mathfrak{h} \cap \mathfrak{s}) \subset (\mathfrak{h} \cap \mathfrak{s}) + \mathfrak{c}$. D'après le lemme 1, on a donc $x \in \mathfrak{c}$, et par suite u se réduit à l'identité sur $\mathfrak{h} \cap \mathfrak{s}$. Ainsi, $\mathfrak{h} \cap \mathfrak{s} = \mathfrak{h} \cap \mathfrak{s}'$.

Remarque. Des exemples simples montrent qu'on peut avoir $\mathfrak{s} \neq \mathfrak{s}'$.

3. Décomposition des sous-espaces de Weyl. Soient $\mathfrak{n}$ une algèbre de Lie nilpotente, ρ une représentation de $\mathfrak{n}$ dans un espace vectoriel V de dimension finie. Considérons les $\xi \in V$ qui, pour tout $n \in \mathfrak{n}$, sont annulés par une puissance convenable de $\rho(n)$. Leur ensemble est un sous-espace V_1 de V. Il existe un sous-espace V_2 de V, stable pour ρ et supplémentaire de V_1, tel que les $\rho(n)\eta$ ($n \in \mathfrak{n}$, $\eta \in V_2$) engendrent le sous-espace V_2.

En particulier, soient $\mathfrak{a}$ une algèbre de Lie, $\mathfrak{n}$ une sous-algèbre nilpotente de $\mathfrak{a}$. Appliquons ce qui précède à la représentation adjointe de $\mathfrak{n}$ dans $\mathfrak{a}$. Le sous-espace V_1 sera alors désigné par $\mathfrak{z}_{\mathfrak{a}}(\mathfrak{n})$. En vertu de la formule de Leibnitz, c'est une sous-algèbre de $\mathfrak{a}$.

LEMME 2. *Soient $\mathfrak{a}$ une algèbre de Lie sur un corps de caractéristique* 0, *$\mathfrak{r}$ son radical, $\mathfrak{h}$ une sous-algèbre de Cartan de $\mathfrak{a}$, et $\mathfrak{t} = \mathfrak{z}_{\mathfrak{a}}(\mathfrak{h} \cap \mathfrak{r})$.*

(i) *$\mathfrak{h}$ est sous-algèbre de Cartan de $\mathfrak{t}$.*

(ii) $\mathfrak{a} = \mathfrak{t} + \mathfrak{r}$

(iii) *Le radical de $\mathfrak{t}$ est $\mathfrak{t} \cap \mathfrak{r}$.*

Démonstration. Comme $\mathfrak{h}$ est une algèbre nilpotente, on a $\mathfrak{h} \subset \mathfrak{t}$, donc $\mathfrak{h}$ est sous-algèbre de Cartan de $\mathfrak{t}$. Considérons la représentation adjointe ρ de $\mathfrak{h} \cap \mathfrak{r}$ dans $\mathfrak{a}$. Il existe une décomposition en somme directe $\mathfrak{a} = \mathfrak{t} + \mathfrak{t}'$, où $\mathfrak{t}$ et $\mathfrak{t}'$ sont stables pour ρ, et où $[\mathfrak{h} \cap \mathfrak{r}, \mathfrak{t}'] = \mathfrak{t}'$. Alors, $\mathfrak{t}' \subset [\mathfrak{r}, \mathfrak{t}'] \subset \mathfrak{r}$, donc $\mathfrak{a} = \mathfrak{t} + \mathfrak{r}$. Soit $\mathfrak{r}' = \mathfrak{t} \cap \mathfrak{r}$, qui est un idéal résoluble de $\mathfrak{t}$. L'égalité $\mathfrak{a} = \mathfrak{t} + \mathfrak{r}$ prouve que $\mathfrak{t}/\mathfrak{r}'$ est isomorphe à $\mathfrak{a}/\mathfrak{r}$, donc semi-simple; ainsi, $\mathfrak{r}'$ est le radical de $\mathfrak{t}$.

PROPOSITION 3. *Soient $\mathfrak{a}$ une algèbre de Lie sur un corps algébriquement clos de caractéristique* 0, *$\mathfrak{r}$ son radical, $\mathfrak{h}$ une sous-algèbre de Cartan de $\mathfrak{a}$. Pour toute racine α de $\mathfrak{a}$ relativement à $\mathfrak{h}$, soit $\mathfrak{a}^\alpha$ le sous-espace correspondant de $\mathfrak{a}$* (sous-espace de Weyl). *Il existe une section de Levi $\mathfrak{s}$ de $\mathfrak{a}$ telle que tout $\mathfrak{a}^\alpha$ (et en particulier $\mathfrak{h} = \mathfrak{a}^0$) soit somme directe de ses intersections avec $\mathfrak{s}$ et $\mathfrak{r}$.*

Démonstration. Soit $\mathfrak{t} = \mathfrak{z}_{\mathfrak{a}}(\mathfrak{h} \cap \mathfrak{r})$. D'après le lemme 2, (i) et (iii), il existe une section de Levi $\mathfrak{s}$ de $\mathfrak{t}$ telle que $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{s}) + (\mathfrak{h} \cap \mathfrak{r})$. D'après le lemme 2, (ii) et (iii), $\mathfrak{s}$ est section de Levi de $\mathfrak{a}$. On va voir que $\mathfrak{s}$ possède les propriétés de la proposition. Pour tout endomorphisme u d'un espace vectoriel V de dimension finie et tout élément λ du corps de base, nous

désignerons par $V(u, \lambda)$ le sous-espace de V formé des éléments annulés par une puissance assez grande de $u - \lambda$. Soit α une racine de $\mathfrak{a}$ relativement à $\mathfrak{h}$. Le sous-espace $\mathfrak{a}^\alpha$ est l'intersection des $\mathfrak{a}(ad_\mathfrak{a} x, \alpha(x))$ quand x parcourt $\mathfrak{h}$. Soient $x_1 \in \mathfrak{h}, x_2 \in \mathfrak{h}$; d'après des résultats connus, $\mathfrak{a}_1 = \mathfrak{a}(ad_\mathfrak{a} x_1, \alpha(x_1))$ et $\mathfrak{a}_2 = \mathfrak{a}(ad_\mathfrak{a} x_2, \alpha(x_2))$ sont stables pour $ad_\mathfrak{a} x_1$ et $ad_\mathfrak{a} x_2$ et, dans $\mathfrak{a}_1 \cap \mathfrak{a}_2$, $ad_\mathfrak{a}(x_1 + x_2)$ admet la seule valeur propre $\alpha(x_1) + \alpha(x_2)$; donc $\mathfrak{a}(ad_\mathfrak{a}(x_1 + x_2), \alpha(x_1 + x_2)) \supset \mathfrak{a}_1 \cap \mathfrak{a}_2$. Ceci prouve que $\mathfrak{a}^\alpha$ est l'intersection des $\mathfrak{a}$ $(ad_\mathfrak{a} x, \alpha(x))$ quand x parcourt un ensemble de générateurs de l'espace vectoriel $\mathfrak{h}$. Il suffit donc de vérifier que $\mathfrak{a}(ad_\mathfrak{a} x, \alpha(x))$ est somme de ses intersections avec $\mathfrak{s}$ et $\mathfrak{r}$, pour $x \in \mathfrak{h} \cap \mathfrak{s}$ et pour $x \in \mathfrak{h} \cap \mathfrak{r}$. C'est immédiat pour $x \in \mathfrak{h} \cap \mathfrak{s}$, car alors $ad_\mathfrak{a} x$ laisse stables $\mathfrak{s}$ et $\mathfrak{r}$. Supposons maintenant $x \in \mathfrak{h} \cap \mathfrak{r}$. Si $\alpha(x) \neq 0$, on a $\mathfrak{a}(ad_\mathfrak{a} x, \alpha(x)) \subset (ad_\mathfrak{a} x)$. $\mathfrak{a} = [x, \mathfrak{a}] \subset \mathfrak{r}$; et, si $\alpha(x) = 0$, on a $\mathfrak{a}(ad_\mathfrak{a} x, \alpha(x)) \supset \mathfrak{t} \supset \mathfrak{s}$, donc $\mathfrak{a}(ad_\mathfrak{a} x, \alpha(x))$ est encore somme de ses intersections avec $\mathfrak{s}$ et $\mathfrak{r}$.

COROLLAIRE. *Soient $\mathfrak{a}$ une algèbre de Lie sur un corps algébriquement clos de caractéristique 0, $\mathfrak{r}$ son radical, $\mathfrak{s}$ une section de Levi de $\mathfrak{a}$, $\mathfrak{k}$ une sous-algèbre de Cartan de $\mathfrak{s}$. Il existe une sous-algèbre de Cartan $\mathfrak{h}$ de $\mathfrak{a}$ telle que*: (1) $\mathfrak{h} = \mathfrak{k} + (\mathfrak{h} \cap \mathfrak{r})$; (2) *tout sous-espace $\mathfrak{a}^\alpha$ (relativement à $\mathfrak{h}$) est somme de ses intersections avec $\mathfrak{s}$ et $\mathfrak{r}$.*

Démonstration. Soit $\mathfrak{h}'$ une sous-algèbre de Cartan de $\mathfrak{a}$ somme de $\mathfrak{k}$ et d'une sous-algèbre de $\mathfrak{r}$. Il existe (prop. 3) une section de Levi $\mathfrak{s}'$ de $\mathfrak{a}$ telle que tout sous-espace $\mathfrak{a}^{\alpha'}$ (relativement à $\mathfrak{h}'$) soit somme de ses intersections avec $\mathfrak{s}'$ et $\mathfrak{r}$. Soit u un automorphisme spécial transformant $\mathfrak{s}'$ en $\mathfrak{s}$. Alors, $u(\mathfrak{h}') = \mathfrak{h}$ est une sous-algèbre de Cartan de $\mathfrak{a}$, et tout sous-espace $\mathfrak{a}^\alpha$ (relativement à $\mathfrak{h}$) est somme de ses intersections avec $\mathfrak{s}$ et $\mathfrak{r}$. Enfin, $\mathfrak{h} \cap \mathfrak{s} = u(\mathfrak{h}' \cap \mathfrak{s}')$ est l'ensemble des éléments de $\mathfrak{s}$ congrus modulo $\mathfrak{r}$ à un élément de $\mathfrak{h}' \cap \mathfrak{s}'$, de sorte que $\mathfrak{h} \cap \mathfrak{s} = \mathfrak{h}' \cap \mathfrak{s} = \mathfrak{k}$. D'où le corollaire.

Remarque. Soient $\mathfrak{a}$ une algèbre de Lie sur un corps de caractéristique 0, $\mathfrak{r}$ son radical, $\mathfrak{h}$ une sous-algèbre de Cartan de $\mathfrak{a}$. Reprenons l'algèbre $\mathfrak{t} = \mathfrak{z}_\mathfrak{a}(\mathfrak{h} \cap \mathfrak{r})$, dont le radical est $\mathfrak{r}' = \mathfrak{t} \cap \mathfrak{r}$ (lemme 2). Nous allons montrer que $\mathfrak{r}'$ *est nilpotent.* Pour cela, nous établirons d'abord le lemme suivant:

LEMME 3. *Soient $\mathfrak{a}$ une algèbre de Lie sur un corps de caractéristique 0, $\mathfrak{r}$ son radical, $\mathfrak{h}$ une sous-algèbre de Cartan de $\mathfrak{a}$. On a $\mathfrak{r} = [\mathfrak{a}, \mathfrak{r}] + (\mathfrak{h} \cap \mathfrak{r})$.*

Démonstration. Soient $\mathfrak{b} = \mathfrak{a}/[\mathfrak{a}, \mathfrak{r}]$, et f l'application canonique de $\mathfrak{a}$ sur $\mathfrak{b}$. Le radical nilpotent de $\mathfrak{b}$ est nul, autrement dit, $\mathfrak{b}$ est réductive. On sait que $f(\mathfrak{h})$ est une sous-algèbre de Cartan de $\mathfrak{b}$ (**(1)**, prop. 17, p. 217), donc contient le centre de $\mathfrak{b}$, c'est-à-dire son radical $f(\mathfrak{r})$. Donc $\mathfrak{r} \subset \mathfrak{h} + [\mathfrak{a}, \mathfrak{r}]$, et par suite $\mathfrak{r} \subset (\mathfrak{h} \cap \mathfrak{r}) + [\mathfrak{a}, \mathfrak{r}]$. L'inclusion opposée est évidente. D'où le lemme.

Ceci posé, tout élément x de $\mathfrak{r}'$ peut se mettre sous la forme $y + z$, où $y \in \mathfrak{h} \cap \mathfrak{r}$ et $z \in [\mathfrak{a}, \mathfrak{r}]$. On a $y \in \mathfrak{t} \cap \mathfrak{r} = \mathfrak{r}'$, donc $z \in \mathfrak{r}'$. L'endomorphisme

$ad_{\mathfrak{a}}z$ est nilpotent (puisque l'idéal $[\mathfrak{a}, \mathfrak{r}]$ de $\mathfrak{a}$ est nilpotent); a fortiori, $ad_{\mathfrak{t}}z$ est nilpotent. D'autre part, $ad_{\mathfrak{t}}y$ est nilpotent par définition de $\mathfrak{t}$. Donc y et z appartiennent au plus grand idéal nilpotent $\mathfrak{n}$ de $\mathfrak{t}$ (**1**, prop. 4, p. 109), et par suite $x \in \mathfrak{n}$. Donc $\mathfrak{n} = \mathfrak{r}'$, ce qui achève la démonstration de notre remarque.

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SECTION THREE

On the Neumann and Dual-adjoint Problems of Generalized Potential Theory

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The two boundary value problems for harmonic differential forms considered in this note each have an infinity of linearly independent eigensolutions. These give rise to an infinite number of integral constraints on the data of the problem. It is shown that these conditions imply certain pointwise relations for a part of the datum functions. This is a new phenomenon in elliptic differential equations—a pointwise restriction on the data of a global problem.

From this analysis we also construct existence proofs for the two problems. The Neumann existence theorem has already been given in **(2)**, but the result for the dual-adjoint problem introduced here is new. Both proofs are based on the existence theorem for the mixed problem in harmonic forms **(3)**, and do not involve anything more than the "topological" properties of harmonic forms.

1. The Neumann problem. The dependent variables are components of a differential form $\phi = \phi_p$ of degree p:

$$\phi = \sum_{i_1<\ldots<i_p} \phi_{i_1\ldots i_p}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

where, since the differentials dx^i anti-commute, we can assume that $\phi_{i_1\ldots i_p}$ is a totally skew-symmetric tensor **(2, 6, 8)**. We here assume that $1 \leqslant p \leqslant N-1$, N being the dimension of the space of the variables x^i. The differential of ϕ is the $(p+1)$-form

$$d\phi = \sum_{i_1<\ldots<i_p} (d\phi_{i_1\ldots i_p}) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

with $d\phi_{i_1\ldots i_p}$ denoting the conventional scalar total differential of the component function. The dual or adjoint form $*\phi$, of degree $N-p$, is formed by replacing all differentials in each term by all those not originally there, in such a way that tensorial invariance is retained. For this we require a Riemann metric which is assumed to be C^∞ and positive definite **(6)**. The co-differential is $\delta\phi = (-1)^{Np+p+1} * d * \phi$ and has degree $p-1$. The relations

$$d \cdot d\phi \equiv 0, \qquad **\phi = (-1)^{Np+p}\phi, \qquad \delta \cdot \delta\phi = 0$$

hold, the last being an obvious consequence of the first two. If $d\phi = 0$, ϕ is said to be closed, and if $\phi = d\chi$, derived; a derived form is closed. Dually we have coclosed and coderived forms. Stokes formula for p-forms is

$$\int_{C_{p+1}} d\phi = \int_{bC_{p+1}} \phi,$$

where C_{p+1} is a chain of dimension $p + 1$ and bC_{p+1} its boundary. A closed form has equal integrals over homologous cycles—these are the periods for the given homology class. From Stokes formula can be derived the Gauss formula

$$(d\phi,\chi) - (\phi, d\chi) = \int_B \phi \wedge * \chi$$

where if α and β have the same degree p we write

$$(\alpha, \beta) = \int_M \alpha \wedge * \beta.$$

The square integral $N(\alpha) = (\alpha, \alpha)$ is positive definite.

A harmonic form satisfies the generalized Laplace equation

$$\Delta\phi = (d\,\delta + \delta\,d)\,\phi = 0 \tag{1.1}$$

which in a flat space implies that each component of ϕ is a harmonic function. A harmonic field satisfies the stronger equations

$$d\phi = 0, \quad d\delta = 0. \tag{1.2}$$

In these problems we consider a compact domain M bounded by a smooth surface B of dimension $N - 1$, and assign certain subforms, invariantly defined on B, which contain normal derivatives of components of ϕ. If B has the local equation $x^N = 0$, we write **(5)**

$$\phi = t\phi + n\phi = t\phi + \tilde{n}\phi \wedge dx^N,$$

where $t\phi$ and $\tilde{n}\phi$ do not contain dx^N and so define tangential and normal components. The data of the Neumann problem are $t\delta\phi$ and $nd\phi$ **(2)**, and we assume Hölder continuity for the datum components. The existence theorem now states that there exists a harmonic form ϕ with given values of $t\delta\phi$ and $nd\phi$ if and only if these values satisfy

$$\int_B (\tau \wedge * d\phi - \delta\phi \wedge * \tau) = 0 \tag{1.3}$$

for every harmonic field τ regular in $M + B$.

There exist non-vanishing datum functions satisfying (1.3) since $t\tau$ and $t*\tau = *n\tau$ are not independent. The relevant existence theorem states that there exists a unique harmonic field having a given admissible tangential boundary value and given periods on $R_p(M_1 B)$ independent relative p-cycles (1, 5). Apart from a finite number of linearly independent eigenforms, the values of $t*\tau$ are determined by those of $t\tau$. Moreover both

$t\tau$ and $t*\tau$ are admissible (1) as tangential boundary values of closed forms in M. The necessary and sufficient conditions for this are, for $t\tau$,

$$d_B t\tau = td\tau = 0 \tag{1.4}$$

implying that $t\tau$ is closed in B, and

$$\int_{bR_{p+1}} t\tau = \int_{R_{p+1}} d\tau = 0, \qquad R_{p+1} \in H_{p+1}(M, B), \tag{1.5}$$

which shows that $t\tau$ has zero period on every p-cycle of B which bounds in M. Similar conditions for $t * \tau$ are easily written down.

Before analyzing (1.3) in detail, we note that if $t\delta\phi$ and $t*d\phi = *n\phi$ are admissible then (1.3) holds. For if $t*d\phi = t\alpha$ where $d\alpha = 0$ in M, then

$$\int_B \tau \wedge *\,d\phi = \int_B \tau \wedge \alpha = \pm \int_B \alpha \wedge *(*\tau) = \pm\,(d\alpha, \tau) \pm (\alpha, \delta\tau) = 0.$$

The other term in (1.3) vanishes likewise. This suggests that apart from the period conditions the integrals in (1.3) depend on $d_B t*d\phi$ and $d_B t\delta\phi$. Actually if $t\delta\phi$ is admissible and (1.3) holds, then $t*d\phi$ is also admissible since

$$\begin{aligned} N(d\delta\phi) &= (\delta\phi, \delta d\delta\phi) + \int_B \delta\phi \wedge *\,d\delta\phi \\ &= (\delta\phi, \delta\Delta\phi) + \int_B \alpha \wedge *\,d\delta\phi \\ &= 0 + (d\alpha, d\delta\phi) - (\alpha, \delta d\delta\phi) = 0. \end{aligned}$$

Here $t\delta\phi = t\alpha$ with $d\alpha = 0$ in M. Now $d\delta\phi = 0$ implies $\delta d\phi = 0$ which in turn ensures that $d*d\phi = 0$.

2. A lemma on admissible boundary values. We require the following lemma in which a set of integral conditions imply that a certain form defined on B is admissible.

LEMMA. *Let $\xi = \xi_{N-p-1}$ be defined on B. Then*

$$\int_B \tau \wedge \xi = 0 \tag{2.1}$$

for all admissible tangential boundary values τ on B, if and only if ξ is admissible.

Proof. If ξ is admissible there exists a form ρ, with $\xi = t\rho$, $d\rho = 0$ in M. Then, since τ is admissible and can also be extended to a form closed in M,

$$\int_B \tau \wedge \xi = \int_B \tau \wedge \phi = \int_M d(\tau \wedge \phi) = 0$$

by Stokes theorem and the formula

$$d(\tau \wedge \rho) = d\tau \wedge \rho \pm \tau \wedge d\rho.$$

This proves the direct part of the lemma.

To establish the converse part, we must show that ξ satisfies conditions corresponding to (1.4) and (1.5). We show first that ξ is closed in B. From **(4, Theorem 1)** we see that if $t\psi_{p-1}$ is an arbitrary $p-1$ form of class C^1 on B, then there exists a coclosed harmonic form ψ with this tangential boundary value. Set $\tau = d\psi$; then τ is closed and since $\delta d\psi = 0$, τ is coclosed and so is a harmonic field. Now on B we have

$$\begin{aligned} d_B(t\psi \wedge \xi) &= d_B t\psi \wedge \xi \pm t\psi \wedge d_B\xi \\ &= t\tau \wedge \xi \pm t\psi \wedge d_B\xi. \end{aligned}$$

Integrating over the closed surface B we find that the left side yields zero, by Stokes formula. Thus

$$\int_B t\psi \wedge d_B\xi = \int_B t\tau \wedge \xi = 0,$$

by hypothesis; and since $t\psi$ is arbitrary we find that $d_B\xi = 0$ as required.

To complete the proof we must show that ξ has zero periods on the $(N-p-1)$-dimensional cycles bR_{N-p} of B which bound in M. Let Γ^i_{N-p-1} and Γ^i_p $(i = 1, \ldots, R_p(B))$ be a fundamental dual basis for the cycles of B of dimensions $N-p-1$ and p, respectively. Let the intersection matrix of these two systems of cycles be the unit matrix of order $R_p(B)$ **(6, 7)**. We now show that, for a given i, one and only one of the dual pair Γ^i_{N-p-1}, Γ^i_p in B bounds in M. If Γ^i_p bounds in M, then $\Gamma^i_p = bR^i_{p+1}$ for a suitable non-trivial relative cycle R^i_{p+1} of $M(\text{mod } B)$. The intersection number of Γ^i_{N-p-1} with R^i_{p+1} in M is equal to its intersection number in B with $R^i_{p+1} \cap B = b\,R^i_{p+1} = \Gamma^i_p$ and so is unity. But this implies that Γ^i_{N-p-1} does not bound in M, since if it did the intersection number would be zero. Conversely, if Γ^i_p does not bound in M, then it belongs to a non-zero absolute homology class A^i_p of M. Let R^i_{N-p} be the relative $N-p$-cycle of M dual to A^i_p; the boundary cycle bR^i_{N-p} is a non-trivial cycle $\bar{\Gamma}^i_{N-p-1}$ of B. Now the intersection number in B of $\bar{\Gamma}^i_{N-p-1}$ with Γ^i_p is seen to be unity, while its intersection number with all other cycles Γ^j_p of B $(j \neq i)$ is seen to be zero, in view of the corresponding intersection numbers of R^i_{N-p} with Γ^j_p, the latter being regarded as a cycle in M. Since a homology class is determined by the intersection numbers with the dual classes, it follows that $\bar{\Gamma}^i_{N-p-1}$ is homologous in B to Γ^i_{N-p-1} and this shows that Γ^i_{N-p-1} bounds in M. The assertion stated above is now proved.

Let ξ^i and τ^j denote the period integrals

$$\xi^i = \int_{\Gamma^i_{N-p-1}} \xi, \quad \tau^j = \int_{\Gamma^i_p} \tau.$$

Then by the bilinear formula for closed manifolds **(6, p. 85)**, we have

$$\int_B \tau \wedge \xi = \sum_{i=1}^{R_p(B)} \tau^i \xi^i \tag{2.2}$$

where a simplification due to the particular form of the intersection matrix

has been made. Since τ is admissible, its period is zero on those cycles Γ^i_p which bound in M, hence we may omit these terms from the sum in (2.2). There exist admissible boundary values τ having assigned periods on the remaining cycles Γ^i_p, which do not bound in M. By our remark above, the duals of these Γ^i_p are the cycles Γ^i_{N-p-1} which bound in M, and it follows that if the integral (2.2) vanishes for all admissible τ, then the periods ξ^i over the cycles Γ^i_{N-p-1} which bound in M must be zero. This completes the proof that ξ is admissible.

3. Reduction of the Neumann problem to the mixed problem. The analysis of §§ 1 and 2 enables us to give an alternative existence proof for the Neumann problem. For this purpose we assume the existence proof for the mixed problem wherein values of $t\phi$ and $t\delta\phi$ are given, subject to the conditions

$$\int_B t\delta\phi \wedge *\tau = 0 \tag{3.1}$$

for every harmonic field τ with $t\tau = 0$ on B (3). We note that the number of these conditions is the relative Betti number $R_p(M, B)$.

Thus there exists a solution of the mixed problem

$$\Delta\phi_1 = 0, \quad t\phi_1 = 0, \quad t\delta\phi_1 = t\delta\phi. \tag{3.2}$$

From (3.2) we see that

$$\int_B (\tau \wedge *d\phi_1 - \delta\phi_1 \wedge *\tau) = 0, \qquad d\tau = 0, \delta\tau = 0. \tag{3.3}$$

If now we subtract this equation from the necessary condition

$$\int_B (\tau \wedge *d\phi - \delta\phi \wedge *\tau) = 0 \tag{1.3}$$

of the Neumann problem, and take account of the third relation in (3.2), we find

$$\int_B \tau \wedge *d(\phi - \phi_1) = 0, \qquad d\tau = 0, \delta\tau = 0. \tag{3.4}$$

The lemma of § 2 now shows that

$$t*d\ (\phi - \phi_1) \tag{3.5}$$

is an admissible tangential boundary value.

From Theorem II of **(4)**, which is a corollary of the mixed boundary value theorem, we find that there exists a form χ satisfying

$$\delta d\chi = 0, \delta\chi = 0 \tag{3.6}$$

and

$$t*\chi = 0, t*d\chi = t*\delta(\phi - \phi_1). \tag{3.7}$$

The conditions for this theorem are that each of the two data should be

admissible. They are satisfied since (3.5) is admissible. Finally, the form

$$\bar{\phi} = \chi + \phi_1 \tag{3.8}$$

is a solution of the given Neumann problem, for it is a harmonic form, and

$$t\delta\bar{\phi} = t\delta\chi + t\delta\phi_1 = t\delta\phi,$$
$$t*d\bar{\phi} = t*d\chi + t*d\phi_1 = t*d\phi,$$

from (3.2), (3.6) and (3.7). This completes our alternative existence proof for the Neumann problem.

From the lemma we may now derive the pointwise conditions applicable to the data. Let $t\delta\phi$ be given; then from the orthogonality conditions (1.3) we see that

$$\int_B \delta\phi \wedge *\tau = \int_B \tau \wedge *d\phi \tag{3.9}$$

and if $t\tau$ vanishes then we must have

$$\int_B \delta\phi \wedge *\tau = 0, \qquad d\tau = 0, \delta\tau = 0, t\tau = 0. \tag{3.10}$$

The $R_p(M, B)$ linearly independent harmonic fields with this property are eigenforms of the mixed tensor boundary value problem.

Now any harmonic field on M can be expressed as the sum of one of these eigenforms plus a uniquely determined field τ with vanishing relative periods. We can choose for the tangential boundary value of τ an arbitrary admissible form on B. From (3.9) we observe that the integral $\int_B \tau \wedge *d\phi$ is thus specified when $t\tau$ is given. From the lemma it is seen that $t*d\phi$ must be determined up to an additive admissible form. This is the set of pointwise conditions referred to in the introduction.

To express these conditions we set $\alpha = d\delta\phi = -\delta d\phi$ and note that α is a harmonic field. Since

$$t\alpha = d_B t\delta\phi, \quad \int_{R_p} \alpha = \int_{bR_p} \delta\phi, \tag{3.11}$$

α is determined uniquely by the assigned values of $t\delta\phi$. Now dually we find that

$$d_B t*d\phi = (-1)^p t*\alpha, \quad \int_{bR_{N-p}} t*d\phi = (-1)^p \int_{R_{N-p}} *\alpha, \tag{3.12}$$

and these are the pointwise and integral constraints on the values of $t*d\phi$.

We may now state the Neumann theorem in the following manner.

THEOREM I. *Let values of $t\delta\phi$ satisfying* (3.10) *be given, and let α denote the harmonic field defined by* (3.11). *Then there exists a harmonic form having the given values of $t*d\phi$ if and only if* (3.12) *hold.*

4. The dual-adjoint problem. Consider the boundary data $t\phi$ and $t*d\phi$ for a harmonic form. The dual data $t*\phi$ and $t\delta\phi$ are in this case just the data of the problem adjoint to the given one in the sense of the theory of differential operators. Hence the name. The eigenforms of this problem are in general infinite in number; they are closed harmonic forms ψ with $t\psi = 0$:

$$d\phi = 0,\ d\delta\psi = 0,\ t\psi = 0. \tag{4.1}$$

By the dual of Theorem II of 4, an arbitrary admissible tangential boundary value can be assigned for ψ.

The dual eigenforms ρ satisfy

$$\delta\rho = 0,\ \delta d\rho = 0,\ t*\rho = 0, \tag{4.2}$$

and from them arise the orthogonality conditions

$$\int_B (\phi \wedge *\, d\rho - \rho \wedge *\, d\phi) = 0, \tag{4.3}$$

which follow at once from Green's second formula. The $R_p(M)$ linearly independent harmonic fields τ with $t*\tau = 0$ are dual eigenforms and thus

$$\int_B \tau \wedge *\, d\phi = 0, \qquad d\tau = 0,\ \delta\tau = 0,\ t*\tau = 0. \tag{4.4}$$

We now show that (4.3) are sufficient for the existence of a solution. Let $\tau\phi$ and $t*d\phi$ be data which satisfy (4.3).

In view of (4.4) there exists a solution ϕ_1 of the mixed problem

$$\Delta\phi_1 = 0,\ t\phi_1 = 0,\ t*d\phi_1 = t*d\phi, \tag{4.5}$$

and this form necessarily also satisfies

$$\int_B (\phi_1 \wedge *\, d\rho - \rho \wedge *\, d\phi_1) = 0$$

for all dual eigenforms ρ. Subtracting this from (4.3), we find

$$\int_B (\phi - \phi_1) \wedge *\, d\rho = 0. \tag{4.6}$$

Here $t*d\rho$ can be given any admissible tangential boundary value. The lemma of §2 now shows that $t(\phi - \phi_1)$ is admissible.

This is just the condition required for the existence of a solution χ of

$$d\chi = 0,\ d\delta\chi = 0,\ t\chi = t\ (\phi - \phi_1);\ t\delta\chi = 0, \tag{4.7}$$

as follows from the dual of Theorem II of (4). Setting

$$\bar{\phi} = \phi_1 + \chi, \tag{4.8}$$

we find $\Delta\,\bar{\phi} = 0$ and

$$t\bar{\phi} = t\phi_1 + t\chi = t\phi,$$
$$t*d\bar{\phi} = t*d\phi_1 + t*d\chi = t*d\phi,$$

from (4.5) and (4.7). This proves the existence theorem for this problem under the regularity conditions applicable to the mixed problem.

The orthogonality conditions are here also equivalent to a number of pointwise conditions. To find these, we set $\beta = d\phi$ and note that β is a closed harmonic form, with $t*\beta = t*d\phi$. By Theorem I of (4) this determines β up to an additive harmonic field with vanishing normal part. Since β is derived, its periods on absolute cycles of M vanish and this additional condition determines β uniquely. Then the datum form must satisfy

$$d_B t\phi = t\beta, \quad \int_{bR_{p+1}} t\phi = \int_{R_{p+1}} \beta. \tag{4.9}$$

These are the pointwise conditions which together with (4.4) are equivalent to (4.3).

THEOREM II. *There exists a harmonic form ϕ with given values for $t\phi$, $t*d\phi$ if and only if* (4.3) *holds; or alternatively if and only if* (4.4) *and* (4.9) *hold, where β is the derived harmonic form having $t*\beta = t*d\phi$.*

As a corollary to this result, we note that if $t*d\phi$ is admissible, then by the dual of Corollary I(b) of **(4)** the above form β is coclosed. In this case ϕ satisfies the two equations

$$d\delta\phi = 0, \ \delta d\phi = 0$$

and so is a biharmonic field.

The dual-adjoint boundary value problem is in general not correctly set in the sense that the number of boundary conditions is not equal to the number of components of ϕ. For $p = 0$ we have harmonic functions and there is no eigenfunction of this problem. The dual eigenforms ρ, however, consist of all harmonic functions v and (4.3) is then a well-known relation

$$\int_B \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS = 0.$$

The content of our theorem is then that a harmonic function u with given values and given normal derivative on B exists if and only if the above relation holds for all harmonic functions v. Thus, for instance, values of u are determined up to a constant (an admissible boundary value in this case) if $\partial u/\partial n$ is given. For $p = N$, harmonic forms are $u dV$, where u is a harmonic function. In this case the boundary conditions are empty and any harmonic form is a solution.

These two problems have been reduced to the mixed problem **(4)** because the various other existence theorems used are all consequences of this one. However, the mixed theorem itself includes both the Dirichlet and Neumann problems for harmonic functions as extreme cases ($p = 0$ or N) and so no new connection for harmonic functions will follow from them. For $1 \leqslant p \leqslant N-1$, however, a new type of relationship has been established.

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SECTION THREE

The Riesz-Fischer Completeness Theorem for Function Spaces and Vector Lattices

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1. Introduction. It is now known that the classical Riesz-Fischer completeness theorem for L^2 permits a lattice theoretic proof which applies on the one hand to arbitrary normed function spaces and on the other hand to general normed vector lattices (J. von Neumann **5**, p. 111; M. H. Stone **6**, p. 340; N. Bourbaki **3**, pp. 130, 131; H. W. Ellis and Israel Halperin **4**, p. 579).

In this paper we shall investigate the assumptions on the normed function space or normed vector lattice which are required for the completeness theorem and we shall see in what way these two general situations differ. In particular, we shall prove among other things (Theorems 3.1, 4.2): a normed vector lattice is complete with respect to the norm if it satisfies the following condition: *whenever a sequence of positive elements* u_n *has* $\sum \|u_n\|$ *finite, then the sums* $(u_1 + u_2 + \ldots + u_n)$ *have at least one upper bound.* This sharpens in two respects a recent result of I. Amemiya **(1)** who proved that completeness holds if every increasing[3] norm-bounded sequence of elements has a least upper bound.

2. The function λ and its properties. Let S be a set of points P with a countably additive, non-negative measure $\gamma(E)$, defined for a complemented countably additive family of sets. In this paper, *all sets and functions which we consider will be* γ*-measurable.*

By $u, v, w, \ldots$ we shall always mean (in §§ 2, 3) non-negative real-valued functions defined on S ($u(P) = \infty$ is permitted). We shall consider a function λ, supposed defined for all u, with $0 \leqslant \lambda(u) \leqslant \infty$, and satisfying:

(L1) $\lambda(u) = 0$ *if* $u(P) = 0$ *almost everywhere.*
(L2) $\lambda(u) \leqslant \lambda(v)$ *whenever* $u < v$.[4]
(L3) $\lambda(u_1 + u_2) \leqslant \lambda(u_1) + \lambda(u_2)$.
(L4) $\lambda(k\,u) = k\,\lambda(u)$ *for all* $k > 0$.

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[3]Increasing, as used in this paper, will mean non-decreasing.
[4]$u < v$ means: $u(P) \leqslant v(P)$ for all P.

Assuming (L1) to (L4) we shall discuss the additional possible conditions listed below:[5]

(L5) $\lambda(\sup u_n) = \sup \lambda(u_n)$ *if the* u_n *are increasing in* n.

$(L5')_k$ $\lambda(\sup u_n) \leqslant k \sup \lambda(u_n)$ *if the* u_n *are increasing in* n.

(L5′) $\lambda(\sup u_n) < \infty$ *if* $\sup \lambda(u_n) < \infty$ *and the* u_n *are increasing in* n.

(L5″) $\lambda(\sum u_n) \leqslant \sum \lambda(u_n)$.

(L5‴) $\lambda(\sum u_n) < \infty$ *if* $\sum \lambda(u_n) < \infty$.

THEOREM 2.1.

(i) (L5) *implies* $(L5')_k$ *for all* $k \geqslant 1$ *but* $(L5')_k$ *for fixed* $k > 1$ *need not imply* (L5).

(ii) $(L5')_k$ *for fixed* $k \geqslant 1$ *implies* (L5′) *and* (L5′) *implies* $(L5')_k$ *for some* $k \geqslant 1$.

(iii) (L5′) *implies* (L5″) *but* (L5″) *need not imply* (L5′).

(iv) (L5″) *and* (L5‴) *are equivalent.*

Proof of (i). That (L5) implies $(L5')_k$ for all $k \geqslant 1$ is obvious. The following example shows that $(L5')_k$ need not imply (L5): Let S be the set of all integers $1, 2, \ldots$; for $E \subset S$ let $\gamma(E)$ be the number of integers in E; suppose K positive and fixed; for $u = (u^i; i = 1, 2, \ldots)$ with u^i non-negative real numbers let $\lambda(u) = \sup u^i + K \limsup u^i$. Then $(L5')_k$ holds for any $k \geqslant K + 1$; but (L5) fails for the sequence $u_n = (u_n{}^i)$ with $u_n{}^i = 1$ for $i \leqslant n$ and $= 0$ for $i > n$.

Proof of (ii). Obviously $(L5')_k$ for fixed $k \geqslant 1$ implies (L5′). To prove the rest of (ii), suppose $(L5')_k$ false for *every* k. Then, because of (L4) there exists for each $N = 1, 2, \ldots$, an increasing sequence $u_n(N)$ satisfying:

$$\lambda(\sup u_n(N)) > N; \lambda(u_n(N)) < N^{-2}.$$

Now set $v_n = u_n(1) + u_{n-1}(2) + \ldots + u_1(n)$. Then for every N, $\sup v_n \geqslant \sup u_n(N)$ so $\lambda(\sup v_n) = \infty$; but $\sup \lambda(v_n) < \infty$ and hence (L5′) is also false. Therefore if (L5′) does hold, then $(L5')_k$ must hold for some k, necessarily $\geqslant 1$ (this is the type of argument used by I. Amemiya **(1)**).

Proof of (iii). It follows from (L3) that $(L5')_k$ for any k implies (L5″) and hence (L5′) implies (L5″). To see that (L5″) need not imply (L5′), let S, γ be as in the proof of (i) but now let $\lambda(u) = \sup u^i$ if $\lim u^i = 0$ and let $\lambda(u) = \infty$ otherwise. Then (L5″) holds; for if $u_n = (u_n{}^i)$ and $\sum \lambda(u_n) < \infty$ then $\sum_n (\sup_i u_n{}^i) < \infty$ and $\lim_i (\sum_n u_n{}^i) = 0$ and $\lambda(\sum u_n) \leqslant \sum \lambda(u_n)$. But (L5′) fails for the sequence $u_n = (u_n{}^i)$ with $u_n{}^i = 1$ for $i \leqslant n$, $= 0$ for $i > n$.

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[5] $u = \sup u_n$ means: $u(P) = \sup u_n(P)$ for all P (of course, $\sup u_n$ is uniquely determined for all u_n).

Proof of (iv). We need only show that (L5‴) implies (L5″). Suppose (L5″) false; then for some $\epsilon > 0$ and some sequence u_n

$$(*) \quad \lambda\Big(\sum u_n\Big) > \epsilon + \sum \lambda(u_n).$$

Because of (L3), (*) remains true if for any m, the elements $u_1, \ldots, u_m$ are omitted. Thus (*) holds with ϵ fixed and $\sum \lambda(u_n)$ arbitrarily small. Now, multiplying each u_n by a suitable factor we can determine, for each $N = 1, 2, \ldots$, a sequence $u_n(N)$ satisfying

$$\sum \lambda(u_n(N)) < N^{-2}; \quad \lambda\Big(\sum u_n(N)\Big) > N.$$

Now set $v_n = u_n(1) + u_{n-1}(2) + \ldots + u_1(n)$; then

$$\sum \lambda(v_n) < \sum n^{-2} \text{ and } \lambda\Big(\sum v_n\Big) = \infty.$$

Thus (L5‴) fails if (L5″) fails so (L5‴) implies (L5″).

We shall say u and v are λ-equivalent and write $u = v(\lambda)$ if $\lambda(|u - v|) = 0$. Clearly $u = v(\lambda)$ implies $\lambda(u) = \lambda(v)$.

We shall write $u \leqslant v(\lambda)$ if $u_1 \leqslant v_1$ for some $u_1 = u(\lambda)$ and $v_1 = v(\lambda)$ (this is equivalent to each of $\lambda(u - \min(u, v)) = 0$, $\lambda(\max(u, v) - v) = 0$). Then $u = v(\lambda)$ is equivalent to $u \leqslant v(\lambda)$ and $v \leqslant u(\lambda)$.

By λ-sup u_n we shall mean any v with the properties: (i) for all n, $u_n \leqslant v(\lambda)$ and (ii) if for some w and all n, $u_n \leqslant w(\lambda)$ then $v \leqslant w(\lambda)$.

Clearly, if λ-sup u_n exists at all it is uniquely determined (to within λ-equivalence) and λ-sup $u_n \leqslant \sup u_n$ (λ). That equality need not hold here may be seen from the following example which has: u_n increasing, $\lambda(u_n) = 0$ for all n (hence λ-sup u_n exists and is the zero function), yet $\lambda(\sup u_n) = \infty$. For the example, let S, γ be as in the proof of Theorem 2.1 (i) but let $\lambda(u) = \limsup u^i$ and choose $u_n = (u_n^i)$ with $u_n^i = i$ if $i \leqslant n$, $= 0$ if $i > n$.

We now list additional possible conditions on λ:

$(\mathrm{L5}^{\mathrm{iv}})_k$ $\sum \lambda(u_n) < \infty$ *implies that there exist* $u_n' = u_n(\lambda)$ *with*

$$\lambda\Big(\sum_{n=r}^{\infty} u_n'\Big) \leqslant k \sum_{n=r}^{\infty} \lambda(u_n)$$

for all r.

$(\mathrm{L5}^{\mathrm{v}})_k$ $\sum \lambda(u_n) < \infty$ *implies that there exist* $u_n' = u_n(\lambda)$ *with* $\lambda(\sum u_n') \leqslant k \sum \lambda(u_n)$.

$(\mathrm{L5}^{\mathrm{v}})$ $\sum \lambda(u_n) < \infty$ *implies that there exist* $u_n' = u_n(\lambda)$ *with* $\lambda(\sum u_n') < \infty$.

$(\mathrm{L5}^{\mathrm{v\,i}})$ $\sum \lambda(u_n) < \infty$ *implies that there exists some* u *with* $\lambda(u) < \infty$ *for which*

$$(u_1 + \ldots + u_n) \leqslant u(\lambda)$$

for all n.

THEOREM 2.2. (i) *The conditions* $(\mathrm{L5}^{\mathrm{v}})$ *and* $(\mathrm{L5}^{\mathrm{v\,i}})$ *are equivalent and they are implied by* (L5‴).

(ii) $(\mathrm{L5}^{\mathrm{v}})$ *implies* $(\mathrm{L5}^{\mathrm{v}})_k$ *for some* $k > 1$.

(iii) $(\mathrm{L5}^{\mathrm{v}})_k$ *implies* $(\mathrm{L5}^{\mathrm{iv}})_k$ *and* $(\mathrm{L5}^{\mathrm{iv}})_k$ *for any* k *implies* $(\mathrm{L5}^{\mathrm{v\,i}})$.

Proof of (i). If $\sum \lambda(u_n) < \infty$ and $(u_1 + \ldots + u_n) \leqslant u(\lambda)$ for all n, then set $u_1' = \min(u, u_1)$ and, by induction on n, set $u_{n+1}' = \min(u - (u_1' + \ldots + u_n'), u_{n+1})$; it follows that $(L5^{v\,1})$ implies $(L5^{v})$. Obviously $(L5^{v})$ implies $(L5^{v\,1})$ and $(L5'''')$ implies $(L5^{v})$.

Proof of (ii). If $(L5^{v})_k$ were false for every $k \geqslant 1$, we could obtain, for each N, a sequence $u_n(N)$ with $\sum \lambda(u_n(N)) < N^{-2}$ but $\lambda(\sum u_n'(N)) > N$ for all choices of $u_n'(N) = u_n(N)$ (λ). Then $(L5^{v})$ fails for all $u_n(N)$ rearranged as a single sequence. Hence $(L5^{v})$ implies that $(L5^{v})_k$ must hold for some k.

Proof of (iii). If $(L5^{v})_k$ holds and $\sum \lambda(u_n) < \infty$ let $u_n'(m) = u_n(\lambda)$ with

$$\lambda\Big(\sum_{n=m}^{\infty} u_n'(m)\Big) \leqslant k \sum_{n=m}^{\infty} \lambda(u_n).$$

Then $(L5^{iv})_k$ holds for $u_n' = \min(u_n'(1), \ldots, u_n'(n))$.

Note: If $(L5^{iv})_k$ holds we may replace u_n' by $\min(u_n, u_n')$. After this replacement we will have $u_n' = \theta_n u_n$ with $0 \leqslant \theta_n(P) \leqslant 1$ for all P.

3. Completeness of the $L^{\lambda}(B)$ spaces. Suppose λ satisfies the conditions (L1), (L2), (L3) and (L4) of §2 and suppose B is a fixed Banach space, e.g., the real or complex numbers. For each function $f(P)$, valued in B and with $||f(P)||$ γ-measurable, we define $\lambda(f)$ to be $\lambda(u)$ with $u(P) = ||f(P)||$. We say f and g are λ-equivalent if $||f(P) - g(P)||$ is a γ-measurable function $u(P)$ for which $\lambda(u) = 0$.

Now let F be any linear space of functions f for which $\lambda(f)$ is defined. We shall restrict F by the conditions: (i) if f is in F so is every g which is λ-equivalent to f and (ii) if f_n are all in F and if for some u with $\lambda(u) < \infty$, $(||f_1(P)|| + \ldots + ||f_n(P)||) \leqslant u(P)$ for all n and P then for suitable f_n' which are λ-equivalent to f_n, $\sum f_n'(P)$ is also in F. For example, F could consist of all strongly, i.e. Bochner, measurable functions.

By $L^{\lambda}(B) = L^{\lambda}(B, F)$ we denote the space of all f in F with $\lambda(f) < \infty$ (more precisely, we mean the λ-equivalence classes of such f). Clearly $L^{\lambda}(B)$ is a normed linear space.

THEOREM 3.1. *$L^{\lambda}(B)$ is complete if $(L5^{v\,1})$ holds.*

Proof. Assume $(L5^{v\,1})$ holds and hence $(L5^{iv})_k$ holds for some fixed k. Suppose f_n in F satisfy $\lambda(f_n) < \infty$, $\lambda(f_n - f_m) \to 0$ as $n, m \to \infty$. Select a subsequence g_i from the f_n such that $\sum \lambda(g_{i+1} - g_i) < \infty$. Consider the expression:

$$||g_1(P)|| + \sum_{i=1}^{\infty} ||g_{i+1}(P) - g_i(P)||.$$

Because of $(L5^{iv})_k$ and the Note at the end of §2 we may replace the g_i by λ-equivalents

$$g_i'' = \theta_i(g_i - g_{i-1}) + g_{i-1}''$$

such that each $||g_{i+1} - g_i||$ gets replaced by a λ-equivalent

$$||g_{i+1}'' - g_i''|| = \theta_{i+1}||g_{i+1} - g_1||$$

and such that after this replacement,

$$\lambda\Big(\sum_{i=m}^{\infty}||g_{i+1}-g_i||\Big) \leqslant k\sum_{i=m}^{\infty}\lambda(||g_{i+1}-g_i||) < \infty.$$

From our restrictions on F, there exist λ-equivalents g_i' such that

$$g_1'(P) + \sum_{i=1}^{\infty}(g_{i+1}'(P) - g_i'(P)),$$

i.e. $\lim g_i'(P)$, defines a function $g(P)$ with g in F. Since

$$||g(P)|| \leqslant ||g_1'(P)|| + \sum_{i=1}^{\infty}||g_{i+1}'(P) - g_i'(P)||$$

for all P, therefore $\lambda(g) < \infty$ and $\lambda(g - g_i) \to 0$ as $i \to \infty$. It follows that g is in $L^\lambda(B, F)$ and $\lambda(g - f_n) \to 0$ as $n \to \infty$.

Thus $L^\lambda(B, F)$ is complete with respect to the norm if (L5^{v1}) holds.

THEOREM 3.2 *If B contains at least one non-zero vector f_0 and if F contains uf_0 for all u with $\lambda(u) < \infty$, then $L^\lambda(B)$ is complete if and only if* (L5^{v1}) *holds.*

Proof. We need only show that (L5^{v1}) holds, assuming that $L^\lambda(B)$ is complete. Suppose u_n increasing with $\sum \lambda(u_n) < \infty$. Then the functions $(u_1 + \ldots + u_n)f_0$ are Cauchy-convergent. Hence for some f in $L^\lambda(B)$, $\lambda((u_1 + \ldots + u_n)f_0 - f) \to 0$ as $n \to \infty$; this implies that

$$\lambda(|(u_1 + \ldots + u_n)||f_0|| - ||f||\,|) \to 0$$

as $n \to \infty$. Hence for a suitable v with $\lambda(v) < \infty$, $\lambda(|(u_1 + \ldots + u_n) - v|) \to 0$ as $n \to \infty$. It is easy to see that u_1 is λ-equivalent to $\min(u_1, v) = u_1'$, say, and by induction, u_n is equivalent to $\min(u_{n+1}, v - (u_1' + u_2' + \ldots + u_n')) = u_{n+1}'$, say. Thus, for all n, $u_1 + \ldots + u_n \leqslant v(\lambda)$ and $\lambda(v) < \infty$ so that (L5^{v1}) does hold.

This type of argument shows that if (L5^{v1}) holds, then whenever an increasing sequence u_n is *dominated* (λ), i.e. for a fixed u with $\lambda(u) < \infty$, $u_n \leqslant u(\lambda)$ for all n, the u_n do have a least upper bound (λ), i.e. λ-sup u_n exists.

4. Normed vector lattices and completeness. A function $||z||$ defined for all z in a vector lattice is called a *semi-norm* if $0 \leqslant ||z|| < \infty$ for all z; $||z|| = ||x||$ with $x = |z|$; and when restricted to positive x the following conditions are satisfied:

(V1) $||x|| = 0$ *if* $x = 0$.
(V2) $||x|| \leqslant ||y||$ *whenever* $x \leqslant y$.
(V3) $||x_1 + x_2|| \leqslant ||x_1|| + ||x_2||$.
(V4) $||k\,x|| = k||x||$ *if* $k > 0$.

We shall write $x = y$ (norm) if $||x - y|| = 0$, and $x \leqslant y$ (norm) if $x_1 \leqslant y_1$ for some $x_1 = x$ (norm) and $y_1 = y$ (norm).

The semi-norm is called a norm if, in addition,

(V1′) $||x|| = 0$ *implies* $x = 0$.

Some additional possible conditions on the semi-norm, *restricted to positive* x, are:[6]

(V5) *If* x_n *is increasing in* n *and* $\sup ||x_n|| < \infty$ *then* $\sup x_n$ *exists and* $||\sup x_n|| = \sup ||x_n||$.

$(V5')_k$ *If* x_n *is increasing in* n *and* $\sup ||x_n|| < \infty$ *then* $\sup x_n$ *exists and* $||\sup x_n|| \leqslant k \sup ||x_n||$.

(V5′) *If* x_n *is increasing in* n *and* $\sup ||x_n|| < \infty$ *then* $\sup x_n$ *exists.*

(V5″) *If* $\sum ||x_n|| < \infty$ *then* $\sup (x_1 + \ldots + x_n)$ *exists and* $||\sup (x_1 + \ldots + x_n)|| \leqslant \sum ||x_n||$.

(V5‴) *If* $\sum ||x_n|| < \infty$ *then* $\sup (x_1 + \ldots + x_n)$ *exists.*

$(V5^{iv})_k$ *If* $\sum ||x_n|| < \infty$ *then there exist* $x_n' = x_n$ *(norm) such that* $\sup (x_1' + \ldots + x_n')$ *exists and*

$$||\sup_n (x_r' + \ldots + x_n')|| \leqslant k\Big(\sum_{n=r}^{\infty} ||x_n||\Big)$$

for all r.

$(V5^{v})_k$ *If* $\sum ||x_n|| < \infty$ *then there exist* $x_n' = x_n$ *(norm) such that* $\sup (x_1' + \ldots + x_n')$ *exists and*

$$||\sup (x_1' + \ldots + x_n')|| \leqslant k \sum ||x_n||.$$

$(V5^{v})$ *If* $\sum ||x_n|| < \infty$ *then there exist* $x_n' = x_n$ *(norm) such that* $\sup (x_1' + \ldots + x_n')$ *exists.*

$(V5^{v1})$ *If* $\sum ||x_n|| < \infty$ *then there exists some* x *such that*

$$(x_1 + \ldots + x_n) \leqslant x \text{ (norm)}$$

for each n.

The functions u in §2, which satisfy the condition $\lambda(u) < \infty$ (more precisely, their γ-equivalence[7] classes) are very much like the positive elements in a semi-normed vector lattice, with the (L) conditions as specializations of the corresponding (V) conditions. This is not quite exact, for in general the γ-equivalence classes of u cannot be embedded in a vector lattice.

Nevertheless, with some changes, the arguments of §2 apply.

If we replace λ by norm then:

THEOREM 4.1. *The statements of Theorem* 2.1 and 2.2 *hold if* L *is replaced throughout by* V.

THEOREM 4.2. *The vector lattice is complete with respect to the norm if and only if* $(V5^{v1})$ *holds.*

In the proof of Theorem 4.2 we need the argument: if z_n are such that for some fixed x, $|z_1| + \ldots + |z_n| \leqslant x$, then for some $z_i' = z_i$ (norm),

[6] If the semi-norm is a norm then (V5) and $(V5)^{v}$ coincide, and other simplifications occur among the (V) conditions; a corresponding remark applies in §2 if $\lambda(u) = 0$ implies $u(P) = 0$ for almost all P.

[7] $u = v(\gamma)$ means: $u(P) = v(P)$ for almost all P.

$\lim (z_1' + \ldots + z_n')$ exists[8] and $|\lim (z_1' + \ldots + z_n')| \leqslant x$. But this argument is valid; for $(V5^{iv})_k$ implies that $\lim (|z_1'| + \ldots + |z_n'| + (z_1' + \ldots + z_n'))$ and $\lim (|z_1'| + \ldots + |z_m'|)$ both exist (equal b, a, respectively, say) and hence $\lim (z_1' + \ldots + z_n')$ exists, and equals $b - a$. Since $|z_1' + \ldots + z_n'| \leqslant |z_1'| + \ldots + |z_n'| \leqslant x$ it follows that $|\lim (z_1' + \ldots + z_n')| \leqslant x$.

It is precisely this point in the lattice-theoretic proof of completeness which is treated differently in function spaces (namely, by a pointwise argument) as compared with vector lattices.

It is possible to give a common abstract formulation: We would consider a semi-vector lattice P of elements u, with a function $\lambda(u)$ satisfying $0 \leqslant \lambda(u) \leqslant \infty$. Then for a linear space V of elements f we would assume a function $|f|$ valued in P. For each f we would define $\lambda(f)$ as $\lambda(|f|)$. Then $L^\lambda(V)$ would consist of the λ-equivalence classes of f with $\lambda(f) < \infty$. We omit the details.

[8]Here $\lim z_n = z$ means that $a_n \leqslant z \leqslant b_n$ for some decreasing sequence of b_n and increasing sequence of a_n such that $a_n \leqslant z_n \leqslant b_n$ for all n and $\inf(b_n - a_n) = 0$.

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SECTION THREE

Initial Segments of Positive Semigroups

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I. Introduction. Zassenhaus recently described the system of all geometric angles by a set of axioms **(4)**. To this purpose he characterized abstractly the additive system of all reals, and incidentally also that of all integers, not exceeding a given constant. This note proposes a somewhat simpler set of axioms to describe any initial segment of the positive reals or integers.

Let $\mathfrak{M}$ be a non-empty set with a partial binary operation $+$. Small Roman letters will denote elements of $\mathfrak{M}$, Gothic capitals denote subsets (or complexes) of $\mathfrak{M}$. $\mathfrak{A} + \mathfrak{B}$ is the set of all c for which there are $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ with $c = a + b$. If $\{a\}$ is the set whose sole member is a, $\{a\} + \mathfrak{A}$ and $\mathfrak{A} + \{a\}$ are freely abbreviated as $a + \mathfrak{A}$ and $\mathfrak{A} + a$ respectively.

We ask when $(\mathfrak{M}, +)$ is abstractly identical with an initial segment of the reals or integers. The following postulates are certainly necessary; we shall prove they are also sufficient.

(1) $(a + b) + c = a + (b + c)$ if either side is defined.

(2) $a \neq b$ if and only if $a \in b + \mathfrak{M}$ or $b \in \mathfrak{M} + a$.

(3) If $\mathfrak{A} \subset \mathfrak{M}$ then $\mathfrak{A} + \mathfrak{M}$ is either empty or $\mathfrak{M}$ or $a + \mathfrak{M}$ for some $a \in \mathfrak{M}$.

Some attempt is made to isolate the contributions of these three axioms. For background material and further references, see **(1)** and **(2)**.

II. Word classes. Let $\mathfrak{M}$ be a system with a partial binary operation $+$. Consider *words* $a_1 + \ldots + a_n$ of lengths $n \geqslant 0$ with $a_i \in \mathfrak{M}$. Words will be denoted by capital Roman letters; 0 is to be the word of length 0. Juxtaposition of words U, V is conveniently symbolized by $U + V$. If $a + b = c$ in $\mathfrak{M}$, we say that the word $U + a + b + V$ *contracts* to $U + c + V$, the latter *expands* to the former. Two distinct words are said to belong to the same *word-class* if we can go from one to the other by a sequence of contractions and expansions. The class containing the word W will be denoted by (W).

LEMMA 1. *If* $(U) = (U')$ *and* $(V) = (V')$ *then* $(U + V) = (U' + V')$.

Proof. There is a sequence $U = U_1, \ldots, U_n = U'$, where U_i contracts or expands to U_{i+1}. Then also $U_i + V$ contracts or expands to $U_{i+1} +$

V, hence $(U + V) = (U' + V)$. Similarly $(U' + V) = (U' + V')$, whence the result.

THEOREM 1. *If $\mathfrak{M}$ is a set with partial addition, the set $\mathfrak{S}(\mathfrak{M})$ of word classes other than (0) is a semigroup under the addition: $(U) + (V) = (U + V)$. Moreover the mapping $a \to (a)$ is a homomorphism from $\mathfrak{M}$ into $\mathfrak{S}(\mathfrak{M})$.*

Proof. Lemma 1 justifies this definition of addition. Its associativity follows from that of juxtaposition. Because $a + b = c$ in $\mathfrak{M}$ implies $(a) + (b) = (a + b) = (c)$, we have the required homomorphism.

III. Consequences of (1). (1) implies the associativity of complex addition:

(1.1) $\quad (\mathfrak{A} + \mathfrak{B}) + \mathfrak{C} = \mathfrak{A} + (\mathfrak{B} + \mathfrak{C})$ for all $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \subset \mathfrak{M}$.

For both sides in (1.1) consist of all sums $a + b + c$ with $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, $c \in \mathfrak{C}$ which are defined in $\mathfrak{M}$. A sum of n complexes may therefore be written without parentheses.

LEMMA 2. *If $a_1 + \ldots + a_m$ and $b_1 + \ldots + b_n$ belong to the same word class then $\{a_1\} + \ldots + \{a_m\} = \{b_1\} + \ldots + \{b_n\}$.*

Proof. Let the two given words be U and V, and denote the corresponding complexes by U^* and V^*; also define $0^* = \{0\}$. It suffices to consider the case when U contracts to V, say $U = P + a + b + Q$, $V = P + c + Q$, where $a + b = c$ in $\mathfrak{M}$. Then $U^* = P^* + \{a\} + \{b\} + Q^* = P^* + \{c\} + Q^* = V^*$, as was to be shown.

THEOREM 2. *If $\mathfrak{M}$ satisfies (1), the mapping $a \to (a)$ embeds it isomorphically in $\mathfrak{S}(\mathfrak{M})$.*

Proof. To see that the mapping is one to one, let $(a) = (b)$, then, by Lemma 2, $\{a\} = \{b\}$, hence $a = b$. To see that its inverse is a homomorphism, let $(c) = (a) + (b) = (a + b)$, then, by Lemma 2, $\{c\} = \{a\} + \{b\}$, hence $c = a + b$ in $\mathfrak{M}$.

We list for later use the following consequence of Lemma 2, leaving its proof to the reader.

LEMMA 2'. *If $(c) = (U + V)$ then $(U) = (a)$ and $(V) = (b)$, where $c = a + b$ in $\mathfrak{M}$.*

IV. Consequences of (2). (2) is equivalent to the following triplet of statements:

(2.1) Neither $a \in a + \mathfrak{M}$ nor $a \in \mathfrak{M} + a$.

(2.2) $a + \mathfrak{M} = \mathfrak{M} + a$.

(2.3) Either $a = b$ or $a \in b + \mathfrak{M}$ or $b \in a + \mathfrak{M}$.

For clearly (2) breaks up into (2.1) and

(2.4) Either $a = b$ or $a \in b + \mathfrak{M}$ or $b \in \mathfrak{M} + a$.

Now by (2.1) neither $a = a + b$ nor $a \in a + b + \mathfrak{M}$. Hence by (2.4) $a + b \in \mathfrak{M} + a$, so that $a + \mathfrak{M} \subset \mathfrak{M} + a$. Similarly we get the converse of this, hence (2.2). But (2.2) produces (2.3) from (2.4) and conversely.

An important consequence of (2) is the cancellation law:

(2.5) If $a + c = b + c$ or $c + a = c + b$ then $a = b$.

For suppose $c + a = c + b$ but $a \neq b$. Then by (2.3) either $a = b + d$ or $b = a + d$, so that $c + b + d = c + b$ or $c + a = c + a + d$, both contrary to (2.1).

LEMMA 3. *If two words belong to the same word class, then some word reduces to both by repeated contractions.*

Proof. It suffices to show this for the special case when the two given words contract to the same word. For then, in going from one to the other by a sequence of contractions and expansions, one can make all expansions precede all contractions, which leads to the required result. It is easily seen that in the special case of the lemma only two subcases arise.

CASE 1. The given words have the form $U + a + b + V + c' + W$ and $U + c + V + a' + b' + W$ where $a + b = c$ and $a' + b' = c'$ in $\mathfrak{M}$. The word $U + a + b + V + a' + b' + W$ will contract to both given words.

CASE 2. The given words are $U + a + b + V$ and $U + a' + b' + V$, where $a + b = a' + b'$ in $\mathfrak{M}$. We may assume that $a + b$ and $a' + b'$ are different words, then $a \neq a'$, in view of (2.5). By (2.3), $a \in a' + \mathfrak{M}$ or $a' \in a + \mathfrak{M}$, say the latter. Thus $a' = a + c$, hence $a + b = a + c + b'$ in $\mathfrak{M}$, so that $b = c + b'$ by (2.5), and $U + a + c + b' + V$ will contract to both given words.

LEMMA 4. *If* $(U) = (U + V)$ *then* $V = 0$.

Proof. Proof is by induction on the length of U. The result is trivial when $U = 0$. Otherwise let $U = a + W$, so that $(a + W) = (a + W + V)$. By Lemma 3, there is a word $A_1 + W_1 = A_2 + W_2 + V_2$, where A_1 and A_2 reduce to a, W_1 and W_2 to W, and V_2 reduces to V by repeated contractions. If A_1 and A_2 differ in length, say $A_2 = A_1 + X$ with $X \neq 0$, then $(a) = (a + X)$, whence $\{a\} = \{a\} + X^*$ by Lemma 2 [see the proof of Lemma 2 for the notation X^*], so that $a = a + b$ with $b \in X^*$, contrary to (2.1). Therefore $A_1 = A_2$, hence $W_1 = W_2 + V_2$, so that $(W) = (W + V)$. But W is shorter than U, hence $V = 0$, by inductional assumption.

LEMMA 5. *For every* $a \in \mathfrak{M}$ *there is a mapping* $U \to U^a$ *such that* $(a + U) = (U^a + a)$.

Proof. Proof is by induction on the length of U. The result is trivial when $U = 0$. Consider $U = b$ of length 1. By (2.3) and (2.2) either $a = b$ or $b = c + a$ or $a = c + b = b' + c$. Then $(a + b)$ is either $(a + a)$ or $(a + c + a)$ or $(b' + c + b) = (b' + a)$; in any case $(a + b) = (b^a + a)$. Finally let $U = V + b$ have length > 1, then $(a + U) = (a + V + b) = (V^a + a + b) = (V^a + b^a + a) = (U^a + a)$, by inductional assumption.

LEMMA 6. *Given two words U and V, then either $(U) = (V + W)$ or $(V) = (U + W)$ for some W.*

Proof. Proof is by induction on the length of $U + V$. If either $U = 0$ or $V = 0$ the result is trivial. Otherwise put $U = a + P$, $V = b + Q$. Now by (2.3) either $a = b + X$ or $b = a + X$, where X has length $\leqslant 1$. Take the first case, for example, then $U = b + X + P$. Now $X + P + Q$ is shorter than $U + V$, hence by inductional assumption $(X + P) = (Q + W)$ or $(Q) = (X + P + W)$. Adding (b) on the left, we obtain the required result.

THEOREM 3. *If $\mathfrak{M}$ satisfies* (1) *and* (2), *then it is isomorphic with an initial segment of the positive part of a simply ordered group.*

Proof. First assume that addition in $\mathfrak{M}$ is always defined, so that $\mathfrak{M}$ is a semigroup satisfying (2). Following a well-known construction, the differences

$$a - b = \{x, y \mid x + a = y + b\}$$

form a group under the addition

$$(a - b) + (c - d) = \begin{cases} a - d \text{ if } b = c \\ a - (s + d) \text{ if } b = s + c \\ (a + t) - d \text{ if } c = b + t. \end{cases}$$

[For an alternate method of defining addition see **(1)**.]
This group is simply ordered by the relation $>$, where

$$(a - b) > (c - d) \text{ means } \begin{cases} a \in c + M \text{ if } b = d \\ a \in s + c + M \text{ if } b = s + d \\ a + t \in c + M \text{ if } d = b + t. \end{cases}$$

The mapping $a \to a + a - a$ is an isomorphism from $\mathfrak{M}$ onto the positive part of this ordered group. The reader will easily supply the details in this standard argument.

Now allow that addition in $\mathfrak{M}$ is not always defined, then $\mathfrak{S}(\mathfrak{M})$ is a semigroup satisfying (2). For $\mathfrak{S}(\mathfrak{M})$ inherits property (2.1), by Lemma 4 and its tacit dual, property (2.2), by Lemma 5 and its dual, and property (2.3), by Lemma 6. By the above argument, $\mathfrak{S}(\mathfrak{M})$ is therefore isomorphic with the positive part of a simply ordered group $\mathfrak{G}$. The order relation of $\mathfrak{G}$ induces an ordering of $\mathfrak{S}(\mathfrak{M})$, namely $(U) > (V)$ whenever $(U) = (V + W)$ for some non-empty word W.

It remains to be shown that the mapping $a \to (a)$ of Theorem 2 sends $\mathfrak{M}$ into an initial segment of $\mathfrak{S}(\mathfrak{M})$. Suppose otherwise, then there is a $c \in \mathfrak{M}$ and a word class (U) containing no word of length 1, such that $(c) > (U)$. But then $(c) = (U + V)$, for some word V, hence $(U) = (c)$, by Lemma 2′, contrary to assumption.

This completes the proof. Observe that the order relation of G induces an ordering of M, namely $a > b$ whenever $a \in b + M$.

V. Consequences of (3)

THEOREM 4. *If $\mathfrak{M}$ satisfies* (1), (2) *and* (3), *then it is isomorphic with an initial segment of the additive semigroup of positive reals or positive integers.*

Proof. It is well known **(2**, p. 30, exercise 33b**)** that a simply ordered group of order > 1 is isomorphic with the group of reals or integers if and only if it is complete, in the sense that every non-empty bounded subset $\mathfrak{A}$ has a greatest lower bound. We shall prove completeness of the ordered group $\mathfrak{G}$ constructed in the proof of Theorem 3. By a suitable translation we can arrange matters so that $\mathfrak{A}$ has both an element a and a lower bound b in $\mathfrak{M}$, which we here identify with its isomorphic image in $\mathfrak{G}$. Hence we may as well assume that $\mathfrak{A}$ is a non-empty bounded subset of $\mathfrak{M}$. According to (3), we have three possibilities: (i) $\mathfrak{A} + \mathfrak{M}$ empty, (ii) $\mathfrak{A} + \mathfrak{M} = \mathfrak{M}$, (iii) $\mathfrak{A} + \mathfrak{M} = c + \mathfrak{M}$ with $c \in \mathfrak{M}$. In the first case a is the only element of $\mathfrak{A}$, for if a and a' are both in $\mathfrak{A}$, then $a + \mathfrak{M}$ and $\mathfrak{M} + a'$ are both empty, whence $a = a'$ by (2.4). In the second case we have $b \in a' + \mathfrak{M}$ for some $a' \in \mathfrak{A}$, hence $b > a'$ in the ordering of $\mathfrak{M}$ induced by that of $\mathfrak{G}$, contrary to assumption. In the last case we show first that c is a lower bound of $\mathfrak{A}$, and secondly that any $c' > c$ is not a lower bound. For by (2.1), c is not in $c + \mathfrak{M} = \mathfrak{A} + \mathfrak{M}$, hence c does not exceed any member of $\mathfrak{A}$; but if $c' \in c + \mathfrak{M} = \mathfrak{A} + \mathfrak{M}$, then c' does exceed some member of $\mathfrak{A}$. This completes the proof.

It is now easily seen that any $\mathfrak{M}$ satisfying (1) to (3) is isomorphic with an initial segment of the positive reals when $\mathfrak{M} + \mathfrak{M} = \mathfrak{M}$ and of the positive integers when $\mathfrak{M} + \mathfrak{M} \neq \mathfrak{M}$. We leave it as an exercise to obtain categorical sets of postulates for each of the following additive systems:

All positive reals ($\mathfrak{R}^+$).

A proper initial segment of $\mathfrak{R}^+$ without largest element.

An initial segment of $\mathfrak{R}^+$ with largest element.

All positive integers.

All positive integers not exceeding n ($n = 1,2,\ldots$).

Note added in proof. This paper was presented to the Society in June 1955 and received by the Transactions in July 1955. In the meantime a similar set of postulates has been given by F. A. Behrend: *A contribution to the theory of magnitudes and the foundations of analysis*, Math. Z. *63* (1956), pp. 345–362.

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SECTION THREE

Characteristic and Order of a Differentiable Point in Conformal n-Space*

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Introduction. In **(1)** the author gave a definition of conformal differentiability of a point of an arc in conformal n-space and he classified the differentiable points. This classification depended on the intersection and support properties of certain families of tangent $(n-1)$-spheres through such a point and on the nature of its osculating m-spheres, $m = 1, 2, \ldots, n-1$. There were $(3n-1)2^{n-1}$ types altogether and each type was described by an $(n+2)$-tuple of numbers called the characteristic of that type. In **(2)**, the authors restricted their attention to the conformal plane and proved that the sum of the digits of the characteristic of an interior differentiable point provided a lower bound for its cyclic order (cf. **2**, Theorem 1). In the present paper, a similar relation is established for conformal n-space. A corresponding theorem for projective space was proved by Scherk in **(3)**.

1. Prerequisites

1.1 The letters $p, t, P, \ldots$ denote points of real conformal n-space; $S^{(m)}$ denotes an m-sphere. When there is no ambiguity, the superscript $n-1$ will be omitted in the case of $S^{(n-1)}$; thus an $(n-1)$-sphere $S^{(n-1)}$ will usually be denoted by S alone. Such an $(n-1)$-sphere S decomposes the n-space into two open regions, its *interior* S_* and its *exterior* S^* (cf. **1**, § 1). If S is a point $(n-1)$-sphere S_* is void.

1.2 A sequence of points $P_1, P_2, \ldots$ is said to be *convergent* to P if to every $(n-1)$-sphere S with $P \subset S_*$ there corresponds a positive integer $N = N(S)$ such that $P_\nu \subset S_*$ if $\nu > N$. The convergence of m-spheres to a point is defined in a similar fashion.

We call a sequence of $(n-1)$-spheres $S_1, S_2, \ldots$ *convergent* to S ($\neq$ point) if to every pair of points $P \subset S_*$ and $Q \subset S^*$ there corresponds a positive integer $N = N(P, Q)$ such that $P \subset S_{\nu*}$ and $Q \subset S^*_\nu$ for every $\nu > N$.

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Finally, a sequence of m-spheres $S_1^{(m)}$, $S_2^{(m)}$, ... will be called convergent to an m-sphere $S^{(m)}$ if to every $S^{(n-m-1)}$ which links with $S^{(m)}$ there exists a positive integer $N = N[S^{(n-m-1)}]$ such that $S_\nu^{(m)}$ links with $S^{(n-m-1)}$ whenever $\nu > N$, $(m = 1, 2, \ldots, n-2)$.

1.3 An arc A is defined as the continuous image of a real interval. If a sequence of points of this parameter interval converges to a point p, the corresponding sequence of image points is defined to be convergent to the image of p. The same small italics $p, t, \ldots$ denote both the points of the parameter interval and their images on A. *The end-points* (*interior points*) of A are the images of the end-points (interior points) of the parameter interval.

A *neighbourhood* of p on A is the image of a neighbourhood of the parameter p on the parameter interval. If p is an interior point of A, this neighbourhood is decomposed by p into two (open) *one-sided-neighbourhoods*. The images of distinct points of the parameter interval are considered to be different points of A, even though they may coincide in space. The notation $Q \neq P$ will indicate that the points Q and P do not coincide.

1.4 Suppose p is an interior point of A. Then we call p a *point of support* (*intersection*) with respect to an $(n-1)$-sphere S if a sufficiently small neighbourhood of p on A is decomposed by p into two one-sided neighbourhoods which lie in the same region (in different regions) bounded by S. S is then called a *supporting* (*intersecting*) $(n-1)$-sphere of A at p. Thus S supports A at p if $p \not\subset S$. By definition, the point $(n-1)$-sphere p always supports A at p. It can happen that every neighbourhood of p has points in common with S. Then S neither supports nor intersects A at p.

1.5 Let p be a fixed point of an arc A, and let t be a variable point of A. Let $1 \leqslant m < n$. If $p, P_1, \ldots, P_{m+1}$ do not lie on the same $(m-1)$-sphere, then there exists a unique m-sphere $S^{(m)}(P_1, \ldots, P_{m+1}, p)$ through these points. It is convenient to denote this m-sphere by the symbol $S_0^{(m)} = S^{(m)}(P_1, \ldots, P_{m+1}; \tau_0)$; here τ_0 indicates that this m-sphere passes through p. In the following, the m-sphere $S^{(m)}(P_1, \ldots, P_{m+1-r}; \tau_r)$ is defined inductively by means of the conditions $\Gamma_r^{(m)}$ given below; (the τ_r in the symbol $S^{(m)}$ $(P_1, \ldots, P_{m+1-r}; \tau_r)$ indicates that this sphere is a *tangent m-sphere* of the arc A at the point p meeting A $r+1$ times at p). We call A $(m+1)$-*times differentiable at* p if the following sequence of conditions is satisfied (cf. **1**, § **4**).

$\Gamma_r^{(m)}$ $(r = 1, 2, \ldots, m+1)$: *If the parameter t is sufficiently close to but different from the parameter p, then the m-sphere $S_r^{(m)} = S^{(m)}$ $(P_1, \ldots, P_{m+1-r}, t; \tau_{r-1})$ is uniquely defined. It converges if t tends to p. Thus its limit sphere*, which will be denoted by $S_r^{(m)} = S^{(m)}(P_1, \ldots, P_{m+1-r}; \tau_r)$, *will be independent of the way t converges to p* [*condition* $\Gamma_{m+1}^{(m)}$ *reads:* $S^{(m)}$ $(t; \tau_m)$ *exists and converges to* $S_{m+1}^{(m)} = S^{(m)}(\tau_{m+1})$].

$S_0^{(0)}$ denotes a pair of points P, p while $S_1^{(0)}$ denotes p. We call A *once differentiable* if $\Gamma_1^{(1)}$ is satisfied. The point p is called a *differentiable point of* A if A is n times differentiable at p. $\tau_r^{(m)}$ will denote the family of all the $S_r^{(m)}$'s (τ_r will mean $\tau_r^{(n-1)}$). In particular, $\tau_{m+1}^{(m)}$ consists only of $S_{m+1}^{(m)}$, the *osculating m-sphere* of A at p.

Differentiability implies:

(i) $\tau_0^{(m)} \supset \tau_1^{(m)} \supset \ldots \supset \tau_{m+1}^{(m)}$ **(1**, Theorem 3, Corollary 4**)**.

(ii) Let $1 \leqslant m < n$. If $S_{m+1}^{(m)} = p$, then $S_{r+1}^{(r)} = p$ $(r = 0, 1, \ldots, m-1)$ **(1**, Theorem 2, Corollary 4**)**.

(iii) $S^{(m-1)}(P_1, \ldots, P_{m-r}; \tau_r) = \bigcap_P S^{(m)}(P_1, \ldots, P_{m-r}, P; \tau_r)$ $(1 < m < n;$ $1 \leqslant r \leqslant m)$. Also, if $P \not\subset S_r^{(m-1)} = S^{(m-1)}(P_1, \ldots, P_{m-r}; \tau_r)$, then $S^{(m)}(P_1, \ldots, P_{m-r}, P; \tau_r) = S^{(m)}[P, S_r^{(m-1)}]$ **(1**, Theorem 2**)**.

(iv) If $S_m^{(m-1)} = p$, the m-spheres of $\tau_m^{(m)}$ touch at p **(1**, Theorem 3**)**.

(v) All the $(n-1)$-spheres of $\tau_r - \tau_{r+1}$ support A at p, or they all intersect; $r = 0, 1, \ldots, n-1$ **(1**, Theorem 8**)**.

(vi) If $S_{r+1}^{(r)} \neq p$, while $S_r^{(r-1)} = p$, then every $(n-1)$-sphere of $\tau_r - \tau_{r+1}$ supports A at p; $1 \leqslant r \leqslant n-1$ cf. **(1**, Theorem 7**)**.

1.6 In (1), the differentiable points p of A were classified into $(3n-1)\,2^{n-1}$ types according to the behaviour of the $(n-1)$-spheres through p and the nature of its osculating m-spheres. With each differentiable point p was associated a *characteristic* $(a_0, a_1, \ldots, a_n; i)$ **(**cf. **1**, § 8**)** such that $a_0, a_1, \ldots, a_{n-1} = 1$ or 2; $a_n = 1$, 2, or ∞; $i = 1, 2, \ldots, n$. In addition, $a_0 + \ldots + a_r \equiv 0$ or 1 (mod 2) according as the $(n-1)$-spheres of $\tau_r - \tau_{r+1}$ support or intersect A at p $(r = 0, 1, \ldots, n-1)$. Also $a_0 + \ldots + a_n$ is even if $S_n^{(n-1)}$ supports, odd if $S_n^{(n-1)}$ intersects, while $a_n = \infty$ if $S_n^{(n-1)}$ neither supports nor intersects. Finally, i is determined as the largest integer $(\leqslant n)$ for which $S_i^{(i-1)} = p$.

1.7 An arc A is said to be of finite order if A has only a finite number of points in common with any $(n-1)$-sphere. If the least upper bound of these numbers is finite, then it is called the *conformal order* of A and A is said to be of *bounded conformal order*. The order of a point p of A is the minimum of the orders of all the neighbourhoods of p on A.

2. A lower bound for the order of a differentiable point

2.1 The following theorem generalizes the Theorem 1 in **(2)**:

THEOREM. *Let p be a differentiable interior point of an arc A. Suppose that p has the characteristic $(a_0, \ldots, a_n; i)$. Then the conformal order of p is not less than $a_0 + \ldots + a_n$.*

Proof. This theorem is a corollary of the following lemma which is proved in §§ 2.2–2.7.

LEMMA. *If the order of p is bounded, then there exists for every neighbourhood of p an $(n-1)$-sphere arbitrarily close to $S_n^{(n-1)}$ which does not pass*

through p, and which intersects that neighbourhood in not less than $a_0 + \ldots + a_n$ points.

2.2 Let B be an arc of finite order. If an $(n-1)$-sphere S intersects B at t, then every $(n-1)$-sphere sufficiently close to S intersects B in at least one point.

Proof. The end-points of some neighbourhood $M \subset B$ of t lie in different regions with respect to S. Hence they also lie in different regions with respect to any $(n-1)$-sphere S' sufficiently close to S. Since M and S' have only a finite number of points in common, one of them must be an intersection. It is clear that S' will intersect M in an odd number of points.

2.3 In the following discussion, π_n will denote $\tau_n = S_n^{(n-1)}$, the unique osculating $(n-1)$-sphere of p; π_{n-1} will denote the linear pencil τ_{n-1}. For $r = n-2, \ldots, 0$, π_r will denote the linear subset of τ_r composed of those $(n-1)$-spheres of τ_r through the $(n-2)$-sphere $S^{(n-2)}(P_1, \ldots, P_{n-r-1}; \tau_r)$ where $p, P_1, \ldots, P_{n-r-1}$ are mutually distinct points such that $S^{(n-1)}(P_1, \ldots, P_{n-r-1}; \tau_{r+1}) \not\subset \tau_{r+2}$ [cf. § 1.5 (iii)]. Finally, π_{-1} will denote the linear pencil of $(n-1)$-spheres through n mutually distinct points $P_1, P_2, \ldots, P_n$, all of which are different from p, and which are such that $S^{(n-1)}(P_1, \ldots; P_n; \tau_0) \not\subset \tau_1$. Thus $\pi_{n-1}(t)$ is $S(t; \tau_{n-1})$; $\pi_r(t)$ is the unique $(n-1)$-sphere $S(P_1, \ldots, P_{n-r-1}, t; \tau_r)$, $r = n-2, \ldots, 1, 0$; and $\pi_{-1}(t)$ is the $(n-1)$-sphere $S(P_1, \ldots, P_n, t)$.

We assume that $a_n < \infty$ in the case of π_{n-1}; hence if t is sufficiently close to p, $\pi_{n-1}(t) = S(t; \tau_{n-1}) \neq S_n^{(n-1)}$ [cf. § 1.6]. Similarly, by § 1.5 (v), $\pi_r(t) = S(P_1, \ldots, P_{n-r-1}, t; \tau_r) \neq S(P_1, \ldots, P_{n-r-1}; \tau_{r+1})$, $r = n-2, \ldots, 1, 0$ and, finally, $\pi_{-1}(t) = S(P_1, \ldots, P_n, t) \notin \tau_0$.

2.4 Let M be any neighbourhood of p on A. We wish to show that π_r contains $(n-1)$-spheres S_r arbitrarily close to but different from

$$T_{r+1} = \lim_{t \to p} \pi_r(t),$$

which meet M outside p in not less than a_{r+1} points $(r = n-1, \ldots, 1, 0, -1)$. If p has finite order, and if M is small enough, we can even say that there are $(n-1)$-spheres of π_r close to T_{r+1} which meet M outside p with a multiplicity $\geqslant a_{r+1}$ and $\equiv a_{r+1} (\mathrm{mod}\ 2)$, $(r = n-1, \ldots, 0, -1)$.

2.5 Let S_r be an $(n-1)$-sphere of the family π_r, $S_r \neq T_{r+1}$, $(r = n-1, \ldots, 0, -1)$.

We define the regions

$$E^*_r = [S^*_r \cap T^*_{r+1}] \cup [S^*_r \cap T^*_{r+1}]$$

and

$$E^*_r = [S^*_r \cap T^*_{r+1}] \cup [S^*_r \cap T^*_{r+1}].$$

If $r = i = n-1$, S_r and T_{r+1} touch at p [cf. 1.5 (iv) and § 1.6]; thus we can assume, in this case, that $S^*_r \cap T^*_{r+1}$ is void and $E^*_r = S^*_r \cap T^*_{r+1}$.

Similarly, if $r = n - 1$ and $i = n$, then T_{*r+1} is void, E_{*r} reduces to S_{*r} and E^*_r to S^*_r. Let π_{*r} $[\pi^*_r]$ denote the set of those $(n-1)$-spheres of π_r that pass through E_{*r} $[E^*_r]$. Then every $(n-1)$-sphere of π_r except S_r and T_{r+1} belongs either to π_{*r} or to π^*_r. By intersecting π_r with an orthogonal circle, we can construct a one-to-one correspondence between the $(n-1)$-spheres of $\pi_{*r}[\pi^*_r]$ and the points of a real interval, and hence an ordering of the $(n-1)$-spheres in $\pi_{*r}[\pi^*_r]$.

We can choose our neighbourhood M so small that S_r and T_{r+1} have no points in common with the two-one-sided neighbourhoods N and N' into which M is decomposed by p. This follows for $S = S_n^{(n-1)}$ from our assumption $a_n < \infty$, and for the other $(n-1)$-spheres it follows from § 1.5 (v). Thus N $[N']$ lies either entirely in the region E_{*r} or else in E^*_r. In particular, for the case $r = i = n - 1$, T_n and M both lie in $S_{*n-1} \cup p$ or both lie in $S^*_{n-1} \cup p$; cf. § 1.5 (iv):

$$\lim_{t \to p} \pi_{n-1}(t) = \lim_{t \to p} S(t; \tau_{n-1}) = S_n^{(n-1)} = T_n.$$

Let t and t' denote the points of N and N' respectively; thus either all the $(n-1)$-spheres of $\pi_r(t)$ belong to π_{*r} or all of them are in π^*_r. With no loss in generality we may assume that $N \subset (S^*_r \cap T^*_{r+1}) \subset E^*_r$. Then $\pi_r(t)$ belongs to π^*_r for every t. Let $e \subset N$. Then $\pi_r(e)$ is the end $(n-1)$-sphere of a one-sided neighbourhood δ of T_{r+1} in π_r. If t moves from e to p, then $\pi_r(t)$ moves in π^*_r from $\pi_r(e)$ to T_{r+1}. Hence the $(n-1)$-spheres of $\pi_r(t)$ omit none of the $(n-1)$-spheres of δ and, consequently, every $(n-1)$-sphere of δ meets N. Let $S \in \delta$. Thus S lies between $\pi_r(e)$ and

$$T_{r+1} = \lim_{t \to p} \pi_r(t).$$

If t is sufficiently close to p, then t does not belong to S and S will also lie between $\pi_r(e)$ and $\pi_r(t)$. Since $e \not\subset S$, and since the points t and e lie in $S^*_r \cap T^*_{r+1}$, they will also be separated by S.

Let the order of p be finite. Then we may assume that M also is of finite order. In addition, S will meet N in a finite number of points only and at least one of them will be an intersection. Replacing N by the one-sided neighbourhood of p with end-point e, we can even state that S will intersect N in an odd number of points.

Similarly, there exists a one-sided neighbourhood δ' of T_{r+1} in π_r such that each of its $(n-1)$-spheres meets N'. If p has finite order and if N' is sufficiently small, then δ' can be chosen such that each $(n-1)$-sphere of δ' intersects N' in an odd number of points.

2.6 If $a_{r+1} = 1$, then one of the $(n-1)$-spheres S_r and T_{r+1} intersects while the other one supports M at p; therefore $N' \subset E_{*r}$. If $a_{r+1} = 2$, then S_r and T_{r+1} either both intersect or both support; hence $N' \subset E^*_r$. Thus the $(n-1)$spheres $\pi_r(t')$ belong to π_{*r} or to π^*_r according as $a_{r+1} = 1$ or 2. This holds true, in particular, of the $(n-1)$-spheres of the neighbourhoods δ and δ'. Since $\delta \subset \pi^*_r$, it follows that δ and δ' lie on opposite

sides of T_{r+1} or on the same side depending on whether $a_{r+1} = 1$ or 2. This implies our statements in § 2.4.

2.7 The proof of our lemma now follows readily. Obviously, we may assume that the order of p is finite and, in particular, $a_n < \infty$.

We prove our lemma by approximating $T_n = S_n^{(n-1)}$ by an $(n-1)$-sphere S_{n-1} of π_{n-1}, and then S_r by an $(n-1)$-sphere S_{r-1} of π_{r-1}, $r = n-1, n-2, \ldots, 0$, in turn.

Let M_n be a neighbourhood of finite order of p on A. From § 2.4 there exists an $(n-1)$-sphere S_{n-1} of π_{n-1}, close to but different from $T_n = S_n^{(n-1)}$, which intersects M_n in not less than a_n points t_n outside p [here t_m represents one (two) points if $a_m = 1$ $(= 2)$].

We proceed by induction and assume that there is an $(n-1)$-sphere S_r of π_r through $S^{(n-2)}(P_1, \ldots, P_{m-r-1}; \tau_r)$ which intersects M_n in $a_n + \ldots + a_{r+1}$ points $t_n, \ldots, t_{r+1}$; $0 \leqslant r < n$. Construct mutually disjoint neighbourhoods $B_n, \ldots, B_{r+1}$ about the points $t_n, \ldots, t_{r+1}$ and M_r about p. Choose a point $P_{n-r} \subset S_r$, $P_{n-r} \not\subset S^{(n-2)}(P_1, \ldots, P_{n-r-1}; \tau_r)$. Then $S_r = S(P_1, \ldots, P_{n-r}; \tau_r)$ [cf. § 1.5 (iii)]. From § 2.4 there exists an $(n-1)$-sphere S_{r-1} of π_{r-1}, close to but different from S_r, which intersects M_r in not less than a_r points t_r and which also intersects each of the neighbourhoods $B_n, \ldots, B_{r+1}$. For $r = 0$ this yields an $(n-1)$-sphere S of π_{-1}, close to but different from S_0 which intersects M_0 in not less than a_0 points different from p, and which intersects each of the $a_n + \ldots + a_1$ neighbourhoods $B_n, \ldots, B_1$. Altogether, S will lie close to $S_n^{(n-1)}$ and will intersect M in not less than $a_0 + \ldots + a_n$ points, all of which are different from p.

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SECTION THREE

Some Properties of Approximate Inverses of Matrices

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Presented by R. L. JEFFERY, F.R.S.C.

1. Introduction. The problem of computing the inverse of a matrix of large order has been the subject of much investigation, especially since the advent of large-scale high speed electronic computers. The author's attention had been drawn, by W. C. Fraser, to the fact that a good approximate left inverse may very well be a poor approximate right inverse. At Fraser's suggestion, an investigation into the conditions under which this phenomenon occurs was made, and numerical examples were given to illustrate the implications of the results.

The first results of the investigations were given in **(2)**. The basic theorems are the following:

THEOREM A. Let ϵ and k be two arbitrary positive real numbers. For each $n \geqslant 2$ there exist non singular $(n \times n)$ matrices A and X such that $XA\text{-}I$ has all of its elements less than ϵ in absolute value while $AX\text{-}I$ has all of its elements greater in absolute value than k.

THEOREM B. Let ϵ and k be arbitrary positive numbers such that $k/\epsilon > 4$. Let A and X be (2×2) matrices with the properties:

(1) A has non-negative elements and $\det A = 1$.

(2) $XA\text{-}I$ has all its elements of magnitude ϵ.

(3) $AX\text{-}I$ has all its elements of magnitude greater than k.

If x_1 and x_2 are characteristic roots of A then

$$|x_1 - x_2| \geqslant \sqrt{\left(\frac{k}{\epsilon} - 4\right)}$$

Theorem A shows that computationally it is of importance to distinguish between an approximate left inverse and an approximate right inverse, while Theorem B relates this phenomenon to the von Neumann-Goldstine condition number in the case of 2×2 matrices.

In this paper the following problems are considered:

(1) The magnitude of the error produced in solving a system of linear equations using an approximate left inverse.

(2) The effect of the Hotelling-Bodewig iteration on a one-sided approximate inverse.

(3) Generalizations of Theorem B to $n \times n$ matrices and an alternative proof of Theorem A which shows more clearly the relationship of this phenomenon to the separation of the characteristic roots of A. These latter results were given to the author in a private communication from A. S. Householder and are reproduced here because the author considers them more interesting than his original proofs.

2. An illustrative numerical example. Consider the set of equations

$$\begin{aligned} 100\,x - 49.869\,y + 50.127\,z &= 100.258 \\ -198.563\,x + 100\,y - 98.568\,z &= -197.131 \\ 188.665\,x - 93.876\,y + 94.793\,z &= 189.581 \end{aligned}$$

These have an exact solution $x = 1$, $y = 1$, $z = 1$.

The matrix of these equations is

$$A = \begin{pmatrix} 100 & -49.869 & 50.127 \\ -198.563 & 100 & -98.568 \\ 188.665 & -93.876 & 94.793 \end{pmatrix}$$

An approximate right inverse is the matrix

$$B = \begin{pmatrix} 140.354 & 31.613 & -41.081 \\ 140.456 & 31.992 & -40.609 \\ -140.246 & -31.238 & 41.557 \end{pmatrix}$$

$$AB\text{-}I = \begin{pmatrix} .025 & .075 & .100 \\ .125 & .025 & .150 \\ .075 & -.125 & -.050 \end{pmatrix}$$

Using B to solve the original set of equations one obtains $x = 51.428$, $y = 76.414$, $z = -24.436$, an absurd answer but the worst is yet to come!

The Hotelling-Bodewig iteration defined as $B_1 = B(2I\text{-}AB)$ determines as a second approximation to an inverse of A,

$$B_1 = \begin{pmatrix} 135.975 & 15.161 & -61.913 \\ 135.992 & 15.582 & -61.484 \\ -135.952 & -14.744 & 62.345 \end{pmatrix}$$

In fact

$$AB_1\text{-}I = \begin{pmatrix} -.021 & +.009 & -.015 \\ -.011 & +.008 & +.004 \\ +.011 & -.008 & +.005 \end{pmatrix}$$

Using B_1 to solve the original equations yields $x = -1093.70$, $y = -1093.74$, $z = 1095.62$.

Thus improving the inverse has meant a deterioration in the solution! It is interesting to note that if a left inverse $\tilde{B}$ of A had been used to the same order of error in $\tilde{B}A\text{-}I$, the approximate values of x, y, z would have error of about 0.2 while in the second approximation the values of x, y, z

would be correct to three figures. Hence from the point of view of the computer the theorems of the next section have great practical interest.

3. The approximation theorems.

THEOREM 1. Let

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad (i = 1, 2, 3, \ldots, n)$$

be a system of linear equations which will be written in matrix form $Ax = b$ where A is a non-singular matrix. Let B be an approximate left inverse of A such that $BA\text{-}I = E$ where E has all its elements less than ϵ in absolute value. The approximate solution $\bar{x} = B\ b$ is such that $|\bar{x}_i - x_i| < \epsilon(|x_1| + |x_2| + \ldots + |x_n|)$.

Proof.
$$\begin{aligned} \bar{x} - x &= Bb - A^{-1}b \\ &= (B - A^{-1})b \\ &= (BA - I)A^{-1}b = Ex \end{aligned}$$

If $E = (\epsilon_{ij})$ with $|\epsilon_{ij}| < \epsilon$ then

$$|\bar{x}_i - x_i| = \left| \sum_{j=1}^{n} \epsilon_{ij} x_j \right| < \epsilon \sum_{j=1}^{n} |x_j|.$$

Theorem 1 shows that the error in the solution is controlled by the magnitude in the error of the inverse when considered as a left inverse. The error in B as a right inverse is immaterial.

The question of improving an approximate inverse is now taken up. Let B be an approximation to a left inverse of A such that $BA = I + E$ where E has elements of absolute value less than ϵ. Let $B_1 = (2I\text{-}BA)B$. Then B_1 can be taken as a second approximation usually referred to as the Hotelling-Bodewig approximation. $B_1A = 2BA - BABA = 2I + 2E - (I + E)^2 = I - E^2$. Hence $B_1A - I = -E^2$. If the largest element of E is ϵ and $|\epsilon| < 1/2n$ where n is the order of the matrix A, then the magnitude of the largest element of E^2 is less than $|n\epsilon^2| = |n\epsilon|\ |\epsilon| < \frac{1}{2}|\epsilon|$. Hence, B_1 is a better left inverse than B. If B at the same time is a poor right inverse, $AB = I + K$ where the elements of K may be large, then $AB_1 - I = 2AB - ABAB - I = 2(I + K) - (I + K)^2 - I = -K^2$. It may happen that the elements of K^2 are small, but in the next theorem it is shown that A and B may be chosen in such a way that the elements of K^2 are larger than those of K.

THEOREM 2. For any $n \geqslant 3$ there exist matrices A and B with the following properties.

(1) $BA = I + E$ where the elements of E are all less than ϵ in magnitude.

(2) $AB = I + K$ where the elements of K are all greater in magnitude than k.

(3) The matrix $B_1 = B(2I\text{-}AB)$ is such that the matrix $B_1A\text{-}I$ has all

its elements less than the corresponding ones of BA-I while AB_1-I has all its elements greater than those of AB-I. In other words it is possible for the Hotelling-Bodewig iteration to improve a good left inverse and at the same time deteriorate a right inverse.

Proof. As the general case is proved in the same way only the (3×3) case is considered here. Let $\epsilon < 1$ and $k > 1$ be two positive numbers and take E and K to be the matrices.

$$E = \begin{vmatrix} \frac{\epsilon}{60} & \frac{\epsilon}{20} & \frac{\epsilon}{15} \\ \frac{\epsilon}{12} & \frac{\epsilon}{60} & \frac{\epsilon}{10} \\ \frac{\epsilon}{20} & -\frac{\epsilon}{12} & -\frac{\epsilon}{30} \end{vmatrix}$$

$$K = \begin{vmatrix} k & 3k & 4k \\ 5k & k & 6k \\ 3k & -5k & -2k \end{vmatrix}$$

It is easily verified that the matrices λI-E and λI-K both have invariant factors 1, 1, λ^3. Hence there exists a non-singular matrix A such that $A\,E\,A^{-1} = K$. Take $B = (I + E)A^{-1}$. Then $BA = I + E$ and $AB = A(I + E)A^{-1} = I + A\,E\,A^{-1} = I + K$. It is easily verified that E^2 has elements smaller than those of E and K^2 has elements larger than those of K; hence the iteration improves the left inverse but worsens the right.

It may be remarked here that the proof of this theorem contains in it a proof of Theorem A.

Another point of interest may be noted here. While it has been shown that one iteration may improve a left inverse and deteriorate a right inverse, it is not true that this process may be repeated indefinitely. In fact continued iteration ultimately leads to a good right inverse. This follows readily from continuity considerations but can be seen directly as follows. From $K = A\,E\,A^{-1}$, it follows that $K^n = A\,E^n\,A^{-1}$, i.e., the elements of K^n are linear combinations of those of E^n with fixed coefficients. Hence since the elements of E^n approach zero so do those of K^n.

4. The condition of a matrix. Basically the main interest of this paper has been in the sensitivity of the solution of a system of linear equations to the coefficients appearing in the equation. Let A be the matrix (a_{ij}) and A^{-1} the matrix (A_{ij}). The sensitivity of the solution of the system

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad (i = 1, 2, 3, \ldots, n)$$

to the coefficients a_{ij} and b_i may be measured by the magnitudes of

$$\frac{\partial x_i}{\partial a_{jk}} \text{ and } \frac{\partial x_i}{\partial b_j}.$$

These are easily calculated to have the values

$$\frac{\partial x_i}{\partial a_{jk}} = -A_{ij}x_k \text{ and } \frac{\partial x_i}{\partial b_j} = A_{ij}.$$

Thus if A^{-1} has large elements the solution becomes sensitive to small changes in the coefficients.

To measure the "size" of a matrix various norms have been proposed. Roughly what is required of a norm is the property that matrices with small elements should have small norms and if a sequence of products of such matrices has all the elements approach zero then the product of the corresponding norms should also approach zero. For the above and other theoretical reasons the norm of a square matrix A is any real valued function $\phi(A)$ having the following four properties:

(1) $\phi(A) > 0$ if $A \neq 0$.

(2) $\phi(\alpha A) = |\alpha|\phi(A)$ for any number α.

(3) $\phi(A + B) \leqslant \phi(A) + \phi(B)$.

(4) $\phi(AB) \leqslant \phi(A)\,\phi(B)$.

For matrices with real elements the following functions can be shown to have the properties of a norm:

$$N(A) = \{\mathrm{tr}(A^TA)\}^{\frac{1}{2}}, \quad B(A) = n \max_{i,j} |a_{ij}| \text{ where } A \text{ is an } (n \times n) \text{ matrix.}$$

With any norm ϕ we can assign to a matrix A a number called the ϕ-condition, this being a measure of the sensitivity of the solution of a system of linear equations with matrix A to small changes in the elements of A. In view of the remarks made at the beginning of this section a reasonable value of the ϕ-condition would be a number proportional to $\phi(A)\,\phi(A^{-1})$. In fact, it is usual to take for the ϕ-condition number of A a value $f(n)\,\phi(A)\,\phi(A^{-1})$ where $f(n)$ is a function only of the dimension of the matrix A. For both the N-condition and the B-condition $f(n)$ will be given the value $1/n$. Matrices with large condition numbers are referred to as ill conditioned. Condition numbers need not necessarily be defined in terms of norms. In fact the first such number was taken by von Neumann and Goldstine to be $|\lambda_1/\lambda_n|$ where $|\lambda_1|$ and $|\lambda_n|$ are the magnitudes of the largest and smallest characteristic roots of A.

The following proof of Theorem A gives added information on the connection between the phenomenon of the existence of a good-left, poor-right inverse of a matrix, and the separation of its characteristic roots. Let B be an $(n \times n)$ non-singular symmetric matrix and let λ and μ be characteristic roots and let u and v be corresponding characteristic vectors. Then $Bu = \lambda u$, $v^TB = \mu v^T$. Hence

$$v^T B^{-1} = \frac{1}{\mu} v^T.$$

Put $E = uv^T$ and $X = (I + E)B^{-1}$. It is clear that by properly normalizing

u and v the elements of E may be made arbitrarily small then $XB\text{-}I = E$ has its largest element ϵ. Also

$$BX\text{-}I = B(I + E)B^{-1} - I$$

$$= B\,E\,B^{-1} = Buv^{T}B^{-1} = \frac{\lambda}{\mu}uv^{T} = \left(\frac{\lambda}{\mu}\right)E = K.$$

Since it is always possible to find a symmetric matrix B with given positive characteristic roots one can make the elements of $BX\text{-}I$ arbitrarily large by a proper choice of B.

Theorem B is now generalized as follows:

THEOREM 3. Let A and X be $(n \times n)$ matrices and ϵ and k positive numbers such that

(1) All the elements of $XA\text{-}I$ are of magnitude $\leqslant \epsilon$, but at least one element is of magnitude ϵ.

(2) All the elements of $AX\text{-}I$ are of magnitude $\geqslant k$. Then both the N and B condition numbers of A are of magnitude greater than or equal to

$$\frac{1}{n}\,\frac{k}{\epsilon}.$$

Proof. Let $XA\text{-}I = E$ and $AX\text{-}I = K$. Then $B(E) = n\,\epsilon$ and $B(K) \geqslant nk$. Also $N(K) \geqslant nk$ and $N(E) \leqslant n\,\epsilon$.

Now $E = XA\text{-}I$. Hence $A\,E\,A^{-1} = A(XA\text{-}I)A^{-1} = AX\text{-}I = K$. For any norm ϕ it follows that

$$\phi(K) = \phi(A\,E\,A^{-1}) \leqslant \phi(A)\,\phi(E)\,\phi(A^{-1}).$$

This implies that

$$\frac{1}{n}\,\phi(A)\,\phi(A^{-1}) \geqslant \frac{1}{n}\,\frac{\phi(K)}{\phi(E)}.$$

Hence for either the B-condition or the N-condition, the condition number of A is greater than

$$\frac{1}{n}\,\frac{nk}{n\epsilon} = \frac{1}{n}\,\frac{k}{\epsilon}.$$

Two final observations are worth making. If A is a matrix with real elements the matrix $A^{T}A$ is usually at least as ill-conditioned as is A. For the N-condition and the von Neumann Goldstine condition this is always the case as was shown by O. Taussky in **(3)**. It is still true that if $A^{T}A$ is badly ill-conditioned, there exist good approximate left inverses which are poor approximate right inverses.

However since $A^{T}A$ is symmetric all practical methods of inverting such matrices yield approximations which are at each stage symmetric. In such a case the approximations are two-sided and the difficulty of confusing left and right does not arise.

The problem of confusing left and right inverses did not arise in the

past as all practical hand computation methods yield the approximate inverse which is most appropriate to the problem. With the advent of high speed computing machinery, whenever matrix inversion is required it is usual to call in a subroutine previously programmed for the machine. In such cases there is a real danger than the subroutine will yield the wrong approximation.

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SECTION THREE

On Some Properties of Certain Fractional Integrals

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Presented by G. de B. ROBINSON, F.R.S.C.

1. Introduction. In an earlier paper **(5)** we considered the properties of two integrals defined on the spaces $\Lambda(\alpha, p)$ and $\mathbf{M}(\alpha, p)$ of Lorentz **(3)**. If f was in one of these spaces, then the integrals were defined by

$$\xi(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

and

$$\eta(x) = \int_x^\infty \frac{f(t)}{t}\,dt$$

and we showed that under certain conditions on p and α, ξ and η were in the same space as f.

However, these two integrals are special cases, namely $r = 1$, $s = 0$, of Kober's fractional integrals $\mathfrak{I}_{r,s}f$ and $\mathfrak{K}_{r,s}f$ defined by

I $$\mathfrak{I}_{r,s}f(x) = \frac{x^{-r-s}}{\Gamma(r)}\int_0^x (x-t)^{r-1}t^s f(t)\,dt,$$

and

II $$\mathfrak{K}_{r,s}f(x) = \frac{x^s}{\Gamma(r)}\int_x^\infty (t-x)^{r-1}t^{-r-s}f(t)\,dt$$

or by the alternative formulae, which we shall use occasionally,

Ia $$\mathfrak{I}_{r,s}f(x) = \frac{1}{\Gamma(r)}\int_0^1 (1-t)^{r-1}t^s f(xt)\,dt,$$

and

IIa $$\mathfrak{K}_{r,s}f(x) = \frac{1}{\Gamma(r)}\int_1^\infty (t-1)^{r-1}t^{-r-s}f(xt)\,dt.$$

These fractional integrals have been extensively studied—see for example **(2)**.

Our object here is to generalize the results of **(5)** concerning ξ and η to $\mathfrak{I}_{r,s}$ and $\mathfrak{K}_{r,s}$. These generalizations form the content of section two.

In **(5)** we applied our results concerning ξ and η to obtain certain properties of Laplace transforms. Our new results will permit us to obtain similar

*This work was done in part while the author was a Fellow at the 1956 Summer Research Institute of the Canadian Mathematical Congress.

results concerning two integral transformations $\mathfrak{W}_{k,m} f$ and $\mathfrak{M}_{k,m} f$ defined by

III $$\mathfrak{W}_{k,m} f(s) = \int_0^\infty e^{-\frac{1}{2}st} W_{k+\frac{1}{2},m}(st)(st)^{m-\frac{1}{2}} f(t)\, dt,$$

and

IV $$\mathfrak{M}_{k,m} f(s) = \frac{\Gamma(m+k+1)}{\Gamma(2m+1)} \int_0^\infty e^{-\frac{1}{2}st} M_{k+\frac{1}{2},m}(st)(st)^{-m-\frac{1}{2}} f(t)\, dt,$$

where $W_{k,m}(x)$ and $M_{k,m}(x)$ are Whittaker's functions. Both of these transformations reduce to the Laplace transformation for $k = m$, and the first also reduces to the Laplace transformation for $k = -m$. The first of these transformations has been studied by Varma **(7)** and Singh **(6)**, and is closely related to the second transformation of Meijer **(4)**.

A word about notation. If $f(x)$ is defined on $(0, \infty)$, $f^*(x)$ will denote the equimeasurable rearrangement of $|f|$ in decreasing order. Also, if $p > 1$ we take q as defined by the relation $p^{-1} + q^{-1} = 1$, and if $p = 1$ we take $q^{-1} = 0$. Further, if f_1 and f_2 are two positive functions, with Lorentz **(3; § 3, 4)** we use the notation $f_1 \prec f_2$ to mean that for every $a > 0$

$$\int_0^a f_1(x)\, dx \leqslant \int_0^a f_2(x)\, dx.$$

One result which we shall use extensively, and which follows immediately from the definition of $\|f\|_{\mathbf{M}(\alpha,p)}$, is that if e is a measurable subset of $(0, \infty)$ and f is a measurable function,

$$\int_e |f(x)|^p dx \leqslant \{m(e)^\alpha \|f\|_{\mathbf{M}(\alpha,p)}\}^p.$$

2. Main theorems. Theorems one and two deal with $\mathfrak{I}_{r,s}$ while three and four deal with $\mathfrak{K}_{r,s}$.

THEOREM 1. *If*

(1) $$f \in \Lambda(\alpha, p), \quad 0 < \alpha \leqslant 1, p \geqslant 1,$$

(2) $$s > \frac{\alpha}{p} - 1, r > 0$$

then

(i) $\mathfrak{I}_{r,s} f$ *exists for almost all* $x > 0$ *and is in* $\Lambda(\alpha, p)$,

(ii) $$\|\mathfrak{I}_{r,s} f\|_{\Lambda(\alpha,p)} \leqslant \frac{\Gamma(s+1-\alpha p^{-1})}{\Gamma(r+s+1-\alpha p^{-1}} \|f\|_{\Lambda(\alpha,p)}.$$

Proof. By **(2**; Theorem 2**)**, if $f \in L_p(0, \infty)$, $r > 0$, and $s > p^{-1} - 1$, $\mathfrak{I}_{r,s} f(x)$ exists for almost all $x > 0$. But by **(3**; 3.5(6)**)**, $\Lambda(\alpha, p) \subseteq L_{p/\alpha}(0, \infty)$. Hence if $f \in \Lambda(\alpha, p)$, $r > 0$, and $s > \alpha p^{-1} - 1$, $\mathfrak{I}_{r,s} f(x)$ exists for almost all $x > 0$.

It is clear from Ia that $\mathfrak{I}_{r,s} f^*(x)$ is decreasing for $x > 0$. We show next that $(\mathfrak{I}_{r,s} f)^* \prec \mathfrak{I}_{r,s} f^*$.

Clearly, if $t > 0$, and $e \subseteq (0, \infty)$, e measurable,

$$\int_e |f(xt)|\,dx \leqslant \int_0^{m(e)} f^*(xt)\,dx.$$

Also, since $|\Im_{r,s}f|$ and $(\Im_{r,s}f)^*$ are equimeasurable on $(0, \infty)$, for every $a > 0$, there is a set $e \subseteq (0, \infty)$ with $m(e) = a$ such that

$$\int_0^a (\Im_{r,s}f(x))^* dx = \int_e |\Im_{r,s}f(x)|\,dx.$$

Hence, if $a > 0$, we have, on using Fubini's theorem,

$$\int_0^a (\Im_{r,s}f(x))^* dx = \int_e |\Im_{r,s}f(x)|\,dx$$
$$\leqslant \frac{1}{\Gamma(r)} \int_e dx \int_0^1 (1-t)^{r-1} t^s |f(xt)|\,dt$$
$$= \frac{1}{\Gamma(r)} \int_0^1 (1-t)^{r-1} t^s dt \int_e |f(xt)|\,dx \leqslant \frac{1}{\Gamma(r)} \int_0^1 (1-t)^{r-1} t^s dt \int_0^a f^*(xt)\,dx$$
$$= \frac{1}{\Gamma(r)} \int_0^a dx \int_0^1 (1-t)^{r-1} t^s f^*(xt)\,dt = \int_0^a \Im_{r,s}f^*(x)\,dx,$$

and thus $(\Im_{r,s}f)^* \prec \Im_{r,s}f^*$.

Hence by (3; Theorem 3.4.4.),

$$\alpha x^{\alpha-1}(\Im_{r,s}f)^{*p} \prec \alpha x^{\alpha-1}(\Im_{r,s}f^*)^p,$$

and thus

$$(\|\Im_{r,s}f\|_{\Lambda(\alpha,p)})^p = \alpha \int_0^\infty x^{\alpha-1}(\Im_{r,s}f(x))^{*p} dx \leqslant \alpha \int_0^\infty x^{\alpha-1}(\Im_{r,s}f^*(x))^p dx,$$

and it remains to show that this last integral is finite.

For this, if $p > 1$, we have, on using Hölder's inequality,

$$\Im_{r,s}f^*(x) = \frac{x^{-r-s}}{\Gamma(r)} \int_0^x (x-t)^{r-1} t^s f^*(t)\,dt$$
$$\leqslant \frac{x^{-r-s}}{\Gamma(r)} \left\{ \int_0^x (x-t)^{r-1} t^{s-\alpha/p} dt \right\}^{1/q} \left\{ \int_0^x (x-t)^{r-1} t^{s+\alpha/q} f^*(t)^p dt \right\}^{1/p}$$
$$= \frac{x^{-(r+s)/p-\alpha/pq}}{\Gamma(r)} \mathrm{B}\left(r, s+1-\frac{\alpha}{p}\right)^{1/q} \left\{ \int_0^x (x-t)^{r-1} t^{s+\alpha/q} f^*(t)^p dt \right\}^{1/p},$$

and this inequality remains true if $p = 1$ since then $1/q = 0$.

Hence

$$\alpha \int_0^\infty x^{\alpha-1}(\Im_{r,s}f^*(x))^p dx$$
$$\leqslant \left(\mathrm{B}\left(r, s+1-\frac{\alpha}{p}\right)^{p/q} \Big/ \Gamma(r)^p\right) \alpha \int_0^\infty x^{\alpha/p-(r+s)-1} dx \int_0^x (x-t)^{r-1} t^{s+\alpha/q} f^*(t)^p dt$$
$$= \left(\mathrm{B}\left(r, s+1-\frac{\alpha}{p}\right)^{p/q} \Big/ \Gamma(r)^p\right) \alpha \int_0^\infty t^{s+\alpha/q} f^*(t)^p dt \int_t^\infty x^{\alpha/p-(r+s)-1} (x-t)^{r-1} dx$$

$$= \left(\mathrm{B}\left(r, s+1-\frac{\alpha}{p}\right)^{p/q} \Big/ \Gamma(r)^p\right) \alpha \int_0^\infty t^{\alpha-1} f^*(t)^p dt \int_0^1 (1-u)^{r-1} u^{s-\alpha/p} du \qquad \left(u=\frac{t}{x}\right)$$

$$= \left\{\frac{\Gamma(s+1-\alpha p^{-1})}{\Gamma(r+s+1-\alpha p^{-1})} ||f||_{\Lambda(\alpha,p)}\right\}^p < \infty.$$

Hence $\Im_{r,s} f \in \Lambda(\alpha, p)$, and

$$||\Im_{r,s} f||_{\Lambda(\alpha,p)} \leqslant \frac{\Gamma(s+1-\alpha p^{-1})}{\Gamma(r+s+1-\alpha p^{-1})} ||f||_{\Lambda(\alpha,p)}.$$

THEOREM 2. *If*

(1) $f \in \mathrm{M}(\alpha, p)$, $0 \leqslant \alpha \leqslant 1$, $p \geqslant 1$,

(2) $s > -\alpha - 1/q$, $r > 0$,

then

(i) $\Im_{r,s} f$ *exists for almost all* $x > 0$ *and is in* $\mathrm{M}(\alpha, p)$,

(ii) $$||\Im_{r,s} f||_{M(\alpha,p)} \leqslant \frac{\Gamma(s+\alpha+q^{-1})}{\Gamma(r+s+\alpha+q^{-1})} ||f||_{\mathrm{M}(\alpha,p)}.$$

Proof. If $\alpha = 0$, $\mathrm{M}(\alpha, p) = L_p(0, \infty)$ for which, from **(2**; Theorem 2**)** the theorem is known. If $p\,\alpha > 1$, by **(5**, Theorem 1**)** $\mathrm{M}(\alpha, p)$ is empty, and if $p\,\alpha = 1$, by **(5**; Theorem 1**)** $\mathrm{M}(\alpha, p) = L_p(0, \infty)$, in which case the theorem is trivial. Hence, without loss of generality we may assume $0 < \alpha < 1$, $p\alpha < 1$.

The existence of $\Im_{r,s} f$ will follow from the finiteness a.e. of $\Im_{r,s}|f|$. For this, if $p > 1$, we have from Hölder's inequality

$$\Im_{r,s}|f|(x) = \frac{1}{\Gamma(r)} \int_0^1 (1-t)^{r-1} t^s |f(xt)|\, dt$$

$$\leqslant \frac{1}{\Gamma(r)} \left\{\int_0^1 (1-t)^{r-1} t^{s+\alpha-1/p} dt\right\}^{1/q} \left\{\int_0^1 (1-t)^{r-1} t^{s-(p-1)\alpha+1/q} |f(xt)|^p dt\right\}^{1/p}$$

$$= \frac{\mathrm{B}(r, s+\alpha+q^{-1})^{1/q}}{\Gamma(r)^p} \left\{\int_0^1 (1-t)^{r-1} t^{s-(p-1)\alpha+1/q} |f(xt)|^p dt\right\}^{1/p}.$$

This inequality remains true if $p = 1$, since then $1/q = 0$.

Hence if e is a subset of $(0, \infty)$ of finite measure,

$$\int_e \Im_{r,s}|f|(x)^p dx \leqslant \frac{\mathrm{B}(r, s+\alpha+q^{-1})^{p/q}}{\Gamma(r)^p} \int_e dx \int_0^1 (1-t)^{r-1} t^{s-(p-1)\alpha+1/q} |f(xt)|^p dt$$

$$= \frac{\mathrm{B}(r, s+\alpha+q^{-1})^{p/q}}{\Gamma(r)^p} \int_0^1 (1-t)^{r-1} t^{s-(p-1)\alpha+1/q} dt \int_e |f(xt)|^p dx$$

$$\leqslant \frac{\mathrm{B}(r, s+\alpha+q^{-1})^{p/q}}{\Gamma(r)^p} \int_0^1 (1-t)^{r-1} t^{s-(p-1)\alpha+1/q} dt \int_0^{m(e)} f^*(xt)^p dx$$

$$= \frac{\mathrm{B}(r, s+\alpha+q^{-1})^{p/q}}{\Gamma(r)^p} \int_0^1 (1-t)^{r-1} t^{s-(p-1)\alpha+1/q-1} dt \int_0^{tm(e)} f^*(u)^p du \qquad (u = xt)$$

$$\leqslant \{m(e)^{\alpha}||f||_{\mathbf{M}(\alpha,p)}\}^{p} \frac{\mathrm{B}(r, s+\alpha+q^{-1})^{p/q}}{\Gamma(r)^{p}} \int_{0}^{1}(1-t)^{r-1}t^{s+\alpha+1/q-1}dt$$

$$= \left\{m(e)^{\alpha} \frac{\Gamma(s+\alpha+q^{-1})}{\Gamma(r+s+\alpha+q^{-1})}||f||_{\mathbf{M}(\alpha,p)}\right\}^{p} < \infty.$$

Thus $\mathfrak{I}_{r,s}|f|$ is finite a.e. on e, and hence since e was an arbitrary subset of $(0, \infty)$ of finite measure, $\mathfrak{I}_{r,s}|f|$ is finite a.e. on $(0, \infty)$, and thus $\mathfrak{I}_{r,s}f$ exists for almost all $x > 0$. Further we have shown that

$$\left\{m(e)^{-\alpha}\int_{e}(\mathfrak{I}_{r,s}|f|(x))^{p}dx\right\}^{1/p} \leqslant \frac{\Gamma(s+\alpha+q^{-1})}{\Gamma(r+s+\alpha+q^{-1})}||f||_{\mathbf{M}(\alpha,p)},$$

and hence

$$||\mathfrak{I}_{r,s}|f|\,||_{\mathbf{M}(\alpha,p)} = \sup_{e}\left\{m(e)^{-\alpha}\int_{e}(\mathfrak{I}_{r,s}|f|(x))^{p}dx\right.$$

$$\leqslant \frac{\Gamma(s+\alpha+q^{-1})}{\Gamma(r+s+\alpha+q^{-1})}||f||_{\mathbf{M}(\alpha,p)}.$$

Hence $\mathfrak{I}_{r,s}|f| \in \mathbf{M}(\alpha, p)$, and thus since $|\mathfrak{I}_{r,s}f(x)| \leqslant \mathfrak{I}_{r,s}|f|(x)$, $\mathfrak{I}_{r,s}f \in \mathbf{M}(\alpha, p)$, and (ii) holds.

THEOREM 3. *If*

(1) $f \in \Lambda(\alpha, p)$, $0 < \alpha \leqslant 1$, $p \geqslant 1$,

(2) $s > -\alpha/p$, $r > 0$,

then

(i) $\mathfrak{K}_{r,s}f$ *exists for almost all* $x > 0$ *and is in* $\Lambda(\alpha, p)$,

(ii) $||\mathfrak{K}_{r,s}f||_{\Lambda(\alpha,p)} \leqslant \dfrac{\Gamma(s+\alpha p^{-1})}{\Gamma(r+s+\alpha p^{-1})}||f||_{\Lambda(\alpha,p)}$.

Proof. By (2; Theorem 2), if $f \in L_{p}(0, \infty)$, $r > 0$, and $s > -1/p$; $\mathfrak{K}_{r,s}f$ exists for almost all $x > 0$. But by (3; 3.5(6)), $\Lambda(\alpha, p) \subseteq L_{p/\alpha}(0, \infty)$. Hence if $f \in \Lambda(\alpha, p)$, $r > 0$, and $s > -\alpha/p$, $\mathfrak{K}_{r,s}f$ exists for almost all $x > 0$.

It is clear from IIa that $\mathfrak{K}_{r,s}f^{*}(x)$ is decreasing for $x > 0$. We show next that $(\mathfrak{K}_{r,s}f)^{*} \prec \mathfrak{K}_{r,s}f^{*}$.

Clearly, since $|\mathfrak{K}_{r,s}f|$ and $\mathfrak{K}_{r,s}f^{*}$ are equimeasurable on $(0, \infty)$, for every $a > 0$ there is a set $e \subseteq (0, \infty)$ with $m(e) = a$ such that

$$\int_{0}^{a}(\mathfrak{K}_{r,s}f(x))^{*}dx = \int_{e}|\mathfrak{K}_{r,s}f(x)|\,dx.$$

Hence, if $a > 0$, we have

$$\int_{0}^{a}(\mathfrak{K}_{r,s}f(x))^{*}dx = \int_{e}|\mathfrak{K}_{r,s}f(x)|\,dx$$

$$\leqslant \frac{1}{\Gamma(r)}\int_{e}dx\int_{1}^{\infty}(t-1)^{r-1}t^{-r-s}|f(xt)|\,dt$$

$$= \frac{1}{\Gamma(r)}\int_{1}^{\infty}(t-1)^{r-1}t^{-r-s}dt\int_{e}|f(xt)|\,dx$$

$$\leqslant \frac{1}{\Gamma(r)} \int_1^\infty (t-1)^{r-1} t^{-r-s} dt \int_0^a f^*(xt)\, dx$$

$$= \frac{1}{\Gamma(r)} \int_0^a dx \int_1^\infty (t-1)^{r-1} t^{-r-s} f^*(xt)\, dt = \int_0^a \mathfrak{R}_{r,s} f^*(x)\, dx,$$

and thus $(\mathfrak{R}_{r,s} f)^* < \mathfrak{R}_{r,s} f^*$. Hence by **(3**; Theorem 3.4.4.**)**

$$\alpha x^{\alpha-1} (\mathfrak{R}_{r,s} f(x))^{*p} < \alpha x^{\alpha-1} (\mathfrak{R}_{r,s} f^*(x))^p$$

and thus

$$\{||\mathfrak{R}_{r,s} f||_{\Lambda(\alpha,p)}\}^p = \alpha \int_0^\infty x^{\alpha-1} (\mathfrak{R}_{r,s} f(x))^{*p} dx \leqslant \alpha \int_0^\infty x^{\alpha-1} (\mathfrak{R}_{r,s} f^*(x))^p dx,$$

and it remains to show that this last integral is finite.

For this, if $p > 1$, we have on using Hölder's inequality,

$$\mathfrak{R}_{r,s} f^*(x) = \frac{x^s}{\Gamma(r)} \int_x^\infty (t-x)^{r-1} t^{-r-s} f^*(t)\, dt$$

$$\leqslant \frac{x^s}{\Gamma(r)} \left\{ \int_x^\infty (t-x)^{r-1} t^{-(r+s+\alpha/p)} dt \right\}^{1/q} \left\{ \int_x^\infty (t-x)^{r-1} t^{-(r+s)+\alpha/q} f^*(t)^p dt \right\}^{1/p}$$

$$= \frac{x^{s/p-\alpha/pq}}{\Gamma(r)} \mathrm{B}(r, s+\alpha p^{-1})^{1/q} \left\{ \int_x^\infty (t-x)^{r-1} t^{-(r+s)+s/q} f^*(t)^p dt \right\}^{1/p},$$

and this inequality remains true if $p = 1$ since then $1/q = 0$. Hence

$$\alpha \int_0^\infty x^{\alpha-1} (\mathfrak{R}_{r,s} f(x))^p dx$$

$$\leqslant \frac{\mathrm{B}(r, s+\alpha p^{-1})^{p/q}}{\Gamma(r)^p} \alpha \int_0^\infty x^{\alpha/p+s-1} dx \int_x^\infty (t-x)^{r-1} t^{-(r+s)+\alpha/q} f^*(t)^p dt$$

$$= \frac{\mathrm{B}(r, s+\alpha p^{-1})^{p/q}}{\Gamma(r)^p} \alpha \int_0^\infty t^{-(r+s)+\alpha/q} f^*(t)^p dt \int_0^t (t-x)^{r-1} x^{\alpha/p+s-1} dx$$

$$= \frac{\mathrm{B}(r, s+\alpha p^{-1})^{p/q}}{\Gamma(r)^p} \alpha \int_0^\infty t^{\alpha-1} f^*(t)^p dt \int_0^1 (1-u)^{r-1} u^{\alpha/p+s-1} du \text{ (where } u = tx)$$

$$= \left\{ \frac{\Gamma(s+\alpha p^{-1})}{\Gamma(r+s+\alpha p^{-1})} ||f||_{\Lambda(\alpha,p)} \right\}^p < \infty.$$

Hence, $\mathfrak{R}_{r,s} f \in \Lambda(\alpha, p)$, and

$$||\mathfrak{R}_{r,s} f||_{\Lambda(\alpha,p)} \leqslant \frac{\Gamma(s+\alpha p^{-1})}{\Gamma(r+s+\alpha p^{-1})} ||f||_{\Lambda(\alpha,p)}.$$

THEOREM 4. *If*

(1) $f \in \mathrm{M}(\alpha, p)$, $0 \leqslant \alpha \leqslant 1$, $p \geqslant 1$,

(2) $s > \alpha - 1/p$, $r > 0$,

then

(i) $\mathfrak{R}_{r,s} f$ *exists for almost all* $x > 0$ *and is in* $\mathrm{M}(\alpha, p)$,

(ii) $||\mathfrak{R}_{r,s} f||_{\mathrm{M}(\alpha,p)} \leqslant \dfrac{\Gamma(s-\alpha+p^{-1})}{\Gamma(r+s-\alpha+p^{-1})} ||f|_{\mathrm{M}}|_{(\alpha,p)}$.

Proof. As in Theorem 2, we may suppose $0 < \alpha < 1$, $p \geqslant 1$.

The existence of $\mathfrak{R}_{r,s} f$ will follow from the finiteness a.e. of $\mathfrak{R}_{r,s} |f|$. For this, if $p > 1$, we have from Hölder's inequality

$$\begin{aligned}
\mathfrak{R}_{r,s} |f|(x) &= \frac{1}{\Gamma(r)} \int_1^\infty (t-1)^{r-1} t^{-r-s} |f(xt)|\, dt \\
&\leqslant \frac{1}{\Gamma(r)} \left\{ \int_1^\infty (t-1)^{r-1} t^{\alpha-(r+s+1/p)} dt \right\}^{1/q} \\
&\qquad \left\{ \int_1^\infty (t-1)^{r-1} t^{1/q-(p-1)\alpha-(r+s)} |f(xt)|^p dt \right\}^{1/p} \\
&= \frac{\mathrm{B}(r, s-\alpha+p^{-1})^{1/q}}{\Gamma(r)} \left\{ \int_1^\infty (t-1)^{r-1} t^{1/q-(p-1)\alpha-(r+s)} |f(xt)|^p dt \right\}^{1/p},
\end{aligned}$$

and this inequality remains true for $p = 1$, since then $q^{-1} = 0$.

Hence, if e is any subset of $(0, \infty)$ of finite measure,

$$\begin{aligned}
&\int_e \mathfrak{R}_{r,s} |f|(x)^p dx \\
&\leqslant \frac{\mathrm{B}(r, s-\alpha+p^{-1})^{p/q}}{\Gamma(r)^p} \int_e dx \int_1^\infty (t-1)^{r-1} t^{1/q-(p-1)\alpha-(r+s)} |f(xt)|^p dt \\
&= \frac{\mathrm{B}(r, s-\alpha+p^{-1})^{p/q}}{\Gamma(r)^p} \int_1^\infty (t-1)^{r-1} t^{1/q-(p-1)\alpha-(r+s)} dt \int_e |f(xt)|^p dx \\
&\leqslant \frac{\mathrm{B}(r, s-\alpha+p^{-1})^{p/q}}{\Gamma(r)^p} \int_1^\infty (t-1)^{r-1} t^{1/q-(p-1)\alpha-(r+s)} dt \int_0^{m(e)} f^*(xt)^p dx \\
&= \frac{\mathrm{B}(r, s-\alpha+p^{-1})^{p/q}}{\Gamma(r)^p} \int_1^\infty (t-1)^{r-1} t^{1/q-(p-1)\alpha-(r+s)-1} dt \int_0^{tm(e)} f^*(u)^p du \\
&\qquad\qquad (u = xt) \\
&= (m(e)^\alpha ||f||_{\mathrm{M}(\alpha,p)})^p \frac{\mathrm{B}(r, s-\alpha+p^{-1})^{p/q}}{\Gamma(r)^p} \int_1^\infty (t-1)^{r-1} t^{\alpha-(r+s+1/p)} dt \\
&= \left\{ m(e)^\alpha \frac{\Gamma(s-\alpha+p^{-1})}{\Gamma(r+s-\alpha+p^{-1})} ||f||_{\mathrm{M}(\alpha,p)} \right\}^p < \infty.
\end{aligned}$$

Thus $\mathfrak{R}_{r,s} |f|$ is finite a.e. on $x > 0$, and hence since e was an arbitrary subset of $(0, \infty)$ of finite measure, $\mathfrak{R}_{r,s} |f|$ is finite a.e. on $(0, \infty)$, and thus $\mathfrak{R}_{r,s} f$ exists for almost all $x > 0$. Further, we have shown that

$$\left\{ m(e)^{-\alpha} \int_e (\mathfrak{R}_{r,s} |f|(x))^p dx \right\}^{1/p} \leqslant \frac{\Gamma(s-\alpha+p^{-1})}{\Gamma(r+s-\alpha+p^{-1})} ||f||_{\mathrm{M}(\alpha,p)},$$

so that

$$\begin{aligned}
||\mathfrak{R}_{r,s} |f|\, ||_{\mathrm{M}(\alpha,p)} &= \sup_e \left\{ m(e)^{-\alpha} \int_e (\mathfrak{R}_{r,s} |f|(x))^p dx \right\}^{1/p} \\
&\leqslant \frac{\Gamma(s-\alpha+p^{-1})}{\Gamma(r+s-\alpha+p^{-1})} ||f||_{\mathrm{M}(\alpha,p)}.
\end{aligned}$$

Thus $\mathfrak{R}_{r,s} |f| \in \mathrm{M}(\alpha, p)$, and hence since $|\mathfrak{R}_{r,s} f(x)| \leqslant \mathfrak{R}_{r,s} |f|$, $\mathfrak{R}_{r,s} f \in \mathrm{M}(\alpha, p)$, and (ii) holds.

3. Applications. In **(5**; Theorem 6**)** we showed that if f is in $\Lambda(\alpha, p)$ and $f(s)$ is the Laplace transform of f, then except in the case $\alpha = p = 1$, $s^{-1}f\ (s^{-1}) \in \Lambda(\alpha, p)$. A similar result was proved for $\mathrm{M}(\alpha, p)$. Here we shall generalize these results to the transformations $\mathfrak{W}_{k,m} f$ and $\mathfrak{M}_{k,m} f$. The results for $\mathfrak{W}_{k,m}$ are contained in Theorem 5, and those for $\mathfrak{M}_{k,m}$ in Theorem 6.

THEOREM 5

(a) *If* $f \in \Lambda\ (\alpha, p)$, $0 < \alpha \leqslant 1$, $p \geqslant 1$, *and if* $\frac{1}{2}(\alpha p^{-1} - 1) < m \leqslant -k$, *then except in the case* $p = \alpha = 1$, $\mathfrak{W}_{k,m} f(s)$ *exists for all* $s > 0$, *and* $s^{-1}\, \mathfrak{W}_{k,m}\, f(s^{-1}) \in \Lambda\ (\alpha, p)$.

(b) *If* $f \in \mathrm{M}\ (\alpha, p)$, $0 \leqslant \alpha \leqslant 1$, $p \geqslant 1$ *and if* $-\frac{1}{2}(\alpha + q^{-1}) < m \leqslant -k$, *then except in the case* $p = 1$, $\alpha = 0$, $\mathfrak{W}_{k,m} f(s)$ *exists for all* $s > 0$, *and* $s^{-1}\, \mathfrak{W}_{k,m}\ (s^{-1}) \in \mathrm{M}\ (\alpha, p)$.

Proof of (a). Without loss of generality we may assume $m < -k$ since if $m = -k$, $\mathfrak{W}_{k,m}$ reduces to the Laplace transform, and for this case the theorem is known from **(5**; Theorem 6**)**.

Then, by Theorem 1, $\mathfrak{I}_{-(m+k),2m} f \in \Lambda(\alpha, p)$, and hence if

$$f_{k,m}(s) = \int_0^\infty e^{-st} \mathfrak{I}_{-(m+k),2m} f(t)\, dt, \quad s > 0,$$

then by **(5**; Theorem 6**)**, $s^{-1} f_{k,m}\ (s^{-1}) \in \Lambda(\alpha, p)$.

But if $s > 0$,

$$\begin{aligned} f_{k,m}(s) &= \int_0^\infty e^{-st} \mathfrak{I}_{-(m+k),2m} f(t)\, dt \\ &= \frac{1}{\Gamma(-m-k)} \int_0^\infty e^{-st}\, t^{k-m} dt \int_0^t (t-u)^{-(m+k+1)}\, u^{2m} f(u)\, du \\ &= \frac{1}{\Gamma(-m-k)} \int_0^\infty u^{2m} f(u)\, du \int_u^\infty e^{-st} (t-u)^{-(m+k+1)}\, t^{k-m} dt, \end{aligned}$$

the interchange of integrations being justified by Fubini's theorem since with f, $|f| \in \Lambda(\alpha, p)$, so that by Theorem 1, $\mathfrak{I}_{-(m+k),2m}\, |f| \in \Lambda(\alpha, p)$, and hence from **(5**; Theorem 6**)**

$$\begin{aligned} &\frac{1}{\Gamma(-m-k)} \int_0^\infty e^{-st}\, t^{k-m} dt \int_0^t (t-u)^{-(m+k+1)}\, u^{2m} |f(u)|\, du \\ &= \int_0^\infty e^{-st} \mathfrak{I}_{-(m+k),2m} |f|(t)\, dt < \infty. \end{aligned}$$

Hence

$$\begin{aligned} f_{k,m}(s) &= \frac{1}{\Gamma(-m-k)} \int_0^\infty u^{2m} f(u)\, du \int_u^\infty e^{-st} (t-u)^{-(m+k+1)}\, t^{k-m} dt \\ &= \frac{1}{\Gamma(-m-k)} \int_0^\infty e^{-su} f(u)\, du \int_0^\infty e^{-suv}\, v^{-(m+k+1)}\, (1+v)^{k-m} dv \\ &\qquad\qquad (v = (t-u)/u) \end{aligned}$$

$$= \int_0^\infty e^{-\frac{1}{2}su} W_{k+\frac{1}{2},-m}(su)(su)^{m-\frac{1}{2}} f(u)\,du \qquad \text{(by (1; 6.11.3(18)))}$$

$$= \int_0^\infty e^{-\frac{1}{2}su} W_{k+\frac{1}{2},m}(su)(su)^{m-\frac{1}{2}} f(u)\,du \qquad \text{(by (1; 6.9(8)))}$$

$$= \mathfrak{W}_{k,m} f(s).$$

Hence $\mathfrak{W}_{k,m} f(s)$ exists for all $s > 0$, and $s^{-1}\,\mathfrak{W}_{k,m} f(s^{-1}) \in \Lambda(\alpha, p)$.

Proof of (b). Practically the same as the proof of (a).

THEOREM 6.

(a) *If* $f \in \Lambda(\alpha, p)$, $0 < \alpha \leqslant 1$, $p \geqslant 1$, *and if* $-(m + \alpha/p) < k \leqslant m$ *then except in the case* $\alpha = p = 1$, $\mathfrak{M}_{k,m} f(s)$ *exists for all* $s > 0$, *and* $s^{-1}\,\mathfrak{M}_{k,m} f(s^{-1}) \in \Lambda(\alpha, p)$.

(b) *If* $f \in \mathrm{M}(\alpha, p)$, $0 \leqslant \alpha \leqslant 1$, $p \geqslant 1$, *and if* $-(m + p^{-1} - \alpha) < k \leqslant m$ *then except in the case* $\alpha = 0$, $p = 1$, $\mathfrak{M}_{k,m} f(s)$ *exists for all* $s > 0$, *and* $s^{-1}\,\mathfrak{M}_{k,m} f(s^{-1}) \in \mathrm{M}(\alpha, p)$.

Proof of (a). Without loss of generality we may assume that $k < m$, since if $k = m$, $\mathfrak{M}_{k,m}$ reduces to the Laplace transform, and for this case the theorem is known from **(5; Theorem 6)**.

Then, by Theorem 3, $\mathfrak{K}_{m-k,m+k} f \in \Lambda(\alpha, p)$, and hence if

$$g_{k,m}(s) = \int_0^\infty e^{-st}\,\mathfrak{K}_{m-k,m+k} f(t)\,dt, \quad s > 0,$$

we have from **(5; Theorem 6)** $s^{-1} g_{k,m}(s^{-1}) \in \Lambda(\alpha, p)$.

But if $s > 0$,

$$g_{k,m}(s) = \int_0^\infty e^{-st}\,\mathfrak{K}_{m-k,m+k} f(t)\,dt$$

$$= \frac{1}{\Gamma(m-k)} \int_0^\infty e^{-st} t^{m+k} dt \int_t^\infty (u-t)^{m-k-1} u^{-2m} f(u)\,du$$

$$= \frac{1}{\Gamma(m-k)} \int_0^\infty u^{-2m} f(u)\,du \int_0^u e^{-st} (u-t)^{m-k-1} t^{m+k} dt,$$

the interchange of integrations being justified by Fubini's theorem since with f, $|f| \in \Lambda(\alpha, p)$, so that by Theorem 3, $\mathfrak{K}_{m-k,m+k}|f| \in \Lambda(\alpha, p)$, and hence from **(5; Theorem 6)**.

$$= \frac{1}{\Gamma(m-k)} \int_0^\infty e^{-st} t^{m+k} dt \int_t^\infty (u-t)^{m-k-1} u^{-2m} |f(u)|\,du$$

$$= \int_0^\infty e^{-st}\,\mathfrak{K}_{m-k,m+k} |f|(t)\,dt < \infty.$$

Hence

$$g_{k,m}(s) = \frac{1}{\Gamma(m-k)} \int_0^\infty u^{-2m} f(u)\,du \int_0^u e^{-st} (u-t)^{m-k-1} t^{m+k} dt$$

$$= \frac{1}{\Gamma(m-k)} \int_0^\infty f(u)\,du \int_0^1 e^{-suv} (1-v)^{m-k-1} v^{m+k} dv \qquad (v = ut)$$

$$= \frac{\Gamma(m+k+1)}{\Gamma(2m+1)} \int_0^\infty \Phi(m+k+1, 2m+1; -su)\, f(u)\, du \quad \text{(by (1; 6.5(1)))}$$

$$= \frac{\Gamma(m+k+1)}{\Gamma(2m+1)} \int_0^\infty e^{-su}\, \Phi(m-k, 2m+1; su)\, f(u)\, du \quad \text{(by (1; 6.3(7)))}$$

$$= \frac{\Gamma(m+k+1)}{\Gamma(2m+1)} \int_0^\infty e^{-\frac{1}{2}su}\, \mathrm{M}_{k+\frac{1}{2},m}(su)(su)^{-m-\frac{1}{2}} f(u)\, du \quad \text{(by (1; 6.9(3)))}$$

$$= \mathfrak{M}_{k,m} f(s).$$

Hence $\mathfrak{M}_{k,m} f(s)$ exists for all $s > 0$, and $s^{-1}\, \mathfrak{M}_{k,m} f(s^{-1}) \in \Lambda(\alpha, p)$.

Proof of (b). Practically the same as the proof of (a).

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